HÜSEYİN DEMİR
A life dedicated to problem composing and problem solving

This collection is compiled by
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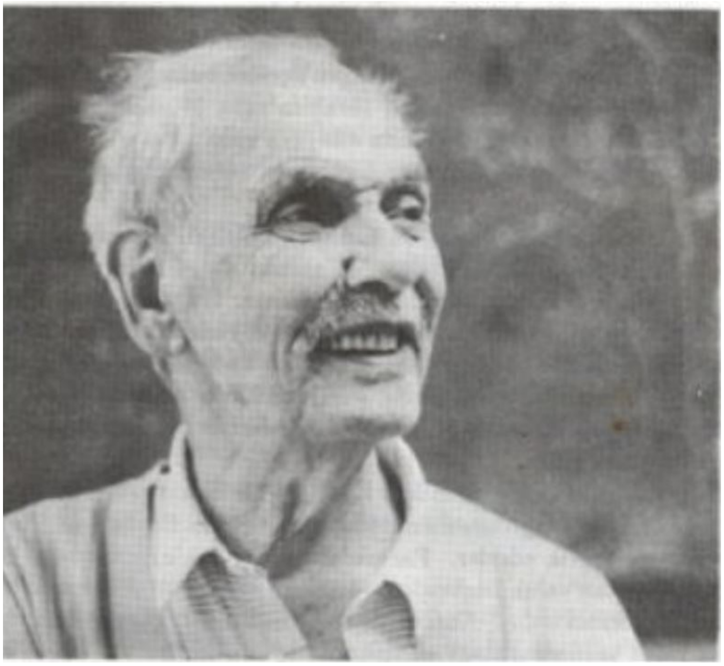
0 Preface



1916-1995

Hüseyin Demir was a prolific problem composer who composed more than one hundred problems in his lifetime. He almost exclusively published in *American Mathematical Monthly* and in *Mathematics Magazine*.

His first publication appeared in 1943 in *American Mathematical Monthly* as "Advanced Problem 4102".

His last publication was in 1993 in *Mathematics Magazine* as the "Solution by the proposer" to his "Proposal 1405" which had appeared the year before.

I have used JSTOR's search engine to find publications of Hüseyin Demir in *American Mathematical Monthly* and *Mathematics Magazine*. Here I not only collected problems proposed by Demir but also the solutions supplied to his problems. In addition to these I also collected published solutions contributed by Demir to other composers's problems.

Hüseyin Demir was an alumnus of Darüşşafaka High School.

Darüşşafaka Cemiyeti was established in 1863 by five young Ottoman gentlemen whose ages were 38, 35, 31, 27 and 24. Sultan Abdülaziz who gave his consent for this establishment was also only 33 years old at the time.

Darüşşafaka is the first non-governmental educational organization in Turkish history. During the last decades of the Empire most muslim Ottoman men who were recruited for the army were lost as the result of long and frequent battles. Thus most families were losing their fathers and their children were then forced to leave their education and start working at *Kapalıçarşı*.

On the other hand non-muslim Ottoman children did not suffer from this mishap. Hence the motto of Darüşşafaka Cemiyeti, at that time was and still today is “Equal opportunity in education”. Children who have lost one of their parents and are financially not able to pursue a proper education are accepted to Darüşşafaka after a competitive entrance exam. Darüşşafaka is a boarding high school and all the daily and educational expenses of students are provided by Darüşşafaka Cemiyeti which is supported to this day by donations. Today Darüşşafaka is recognized as one of the best educational establishments of Turkey.

Salih Zeki, another of our famous mathematicians, was an 1882 alumnus of Darüşşafaka. Hüseyin Demir is 1935 alumnus and according to his telling he read books of Salih Zeki when he was in school. While he was a middle school student at Darüşşafaka he came up with a novel proof of Pythagoras theorem, which is in the genre of “proof without words” and I am reproducing it here.

I found it my responsibility to my school to compile this collection of Hüseyin Demir’s problems. To continue the tradition under which we grew, this collection is meant to be used freely for educational purposes.

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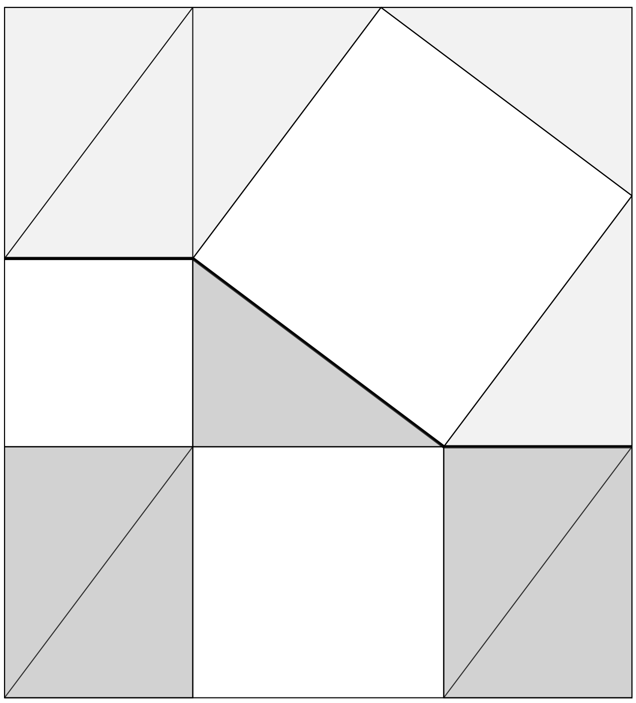

1 Advanced Problems for *MONTHLY*

List of Advanced Problems:

[1] Advanced Problem 4102, *American Mathematical Monthly*, **50**, (1943), 638.
[2] Advanced Problem 4125, *American Mathematical Monthly*, **51**, (1944), 252.
[3] Advanced Problem 4134, *American Mathematical Monthly*, **51**, (1944), 475.
[4] Advanced Problem 4193, *American Mathematical Monthly*, **53**, (1946), 160.
[5] Advanced Problem 4215, *American Mathematical Monthly*, **53**, (1946), 470.
[6] Advanced Problem 4679, *American Mathematical Monthly*, **63**, (1956), 191.
[7] Advanced Problem 4695, *American Mathematical Monthly*, **63**, (1956), 426.
[8] Advanced Problem 4710, *American Mathematical Monthly*, **63**, (1956), 669.
[9] Advanced Problem 4735, *American Mathematical Monthly*, **64**, (1957), 277.
[10] Advanced Problem 4818, *American Mathematical Monthly*, **65**, (1958), 779.

Advanced Problem 4102, *American Mathematical Monthly*, **50**, (1943), 638.

4102. *Proposed by Hüseyin Demir, Columbia University*

Let O and I be respectively the circumcenter and incenter of a given triangle ABC. Let A_0, B_0, C_0 be points taken respectively on BC, CA, AB so that the sums of the algebraic distances of each point to two other sides are equal to a given length l. Prove synthetically that: (1) The points A_0, B_0, C_0 are collinear; (2) The sum of distances to the sides of ABC of points on $A_0B_0C_0$ is the constant l; (3) the line $A_0B_0C_0$ is perpendicular to the line OI.

Advanced Problem 4125, *American Mathematical Monthly*, **51**, (1944), 252.

4125. ***Proposed by Hüseyin Demir, Columbia University***

Prove that

$$\begin{vmatrix} \sin\theta_1 & -e^{-i\theta_1} & 0 & 0 & \cdots 0 & 0 \\ \sin\theta_2 & e^{i\theta_2} & -e^{-i\theta_2} & 0 & \cdots 0 & 0 \\ \sin\theta_3 & 0 & e^{i\theta_3} & -e^{-i\theta_3} & \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sin\theta_n & 0 & 0 & 0 & \cdots 0 & e^{i\theta_n} \end{vmatrix} = \sin(\theta_1+\theta_2+\cdots+\theta_n).$$

Advanced Problem 4134, *American Mathematical Monthly*, **51**, (1944), 475.

4134. ***Proposed by Hüseyin Demir, Columbia University***

Let $C_1^1C_2^1C_3^1$ be the inscribed triangle of a reference triangle $A_1A_2A_3$, and $C_1^2C_2^2C_3^2$ be that of $C_1^1C_2^1C_3^1$, and so on, obtaining a triangle $C_1^nC_2^nC_3^n$ after n steps. Denoting the angles of the nth triangle by C_i^n, prove that

1. $(C_i^n-\pi/3)/(A_i-\pi/3)=(-1)^n2^{-n}$.

2. The limit of the direction of $C_2^nC_3^n$ as $n\to\infty$, is the direction of one of the trisectrices of the angle $(A_2A_3,\ C_2^1C_3^1)$, and from that observe a method of trisecting an angle by ruler and compass in infinitely many steps.

Advanced Problem 4193, *American Mathematical Monthly*, **53**, (1946), 160.

4193. ***Proposed by Hüseyin Demir, Columbia University***

If on the sides of an arbitrary pentagon $A_1A_2A_3A_4A_5$ the triangles $B_iA_{i+2}A_{i+3}$ (with indices reduced mod 5) are constructed such that $B_iA_{i+2}\|A_iA_{i+1}$, and $B_iA_{i+3}\|A_iA_{i+4}$, then the lines A_iB_i concur in a point C.

Advanced Problem 4215, *American Mathematical Monthly*, **53**, (1946), 470.

4215. *Proposed by Hüseyin Demir, Columbia University*

Prove that the Hermite polynomials defined as follows

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2},$$

have the property

$$n! \sum_{p=0}^{n} \frac{H_p^2(x)}{p!} = H_{n+1}^2(x) - H_n(x)H_{n+2}(x).$$

Advanced Problem 4679, *American Mathematical Monthly*, **63**, (1956), 191.

4679. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

If $A_1A_2A_3A_4A_5$ is a cyclic pentagon and if Ω_{ij} denotes the orthopole of the line A_iA_j with respect to the triangle formed by the remaining three vertices, then prove that the ten points Ω_{ij} all lie on a circle.

Advanced Problem 4695, *American Mathematical Monthly*, **63**, (1956), 426.

4695. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that if in a cyclic quadrangle the Simson line of one vertex with respect to the triangle formed by the other three is perpendicular to the Euler line of that triangle, then the same property holds for the other vertices of the quadrangle.

Advanced Problem 4710, *American Mathematical Monthly*, **63**, (1956), 669.

4710. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that if in a complete quadrangle inscribed in a circle (O) one pair of opposite sides are isotomic lines with respect to a triangle inscribed in (O), then the remaining pairs of opposite sides are also isotomic lines with respect to the same triangle.

Advanced Problem 4735, *American Mathematical Monthly*, **64**, (1957), 277.

4735. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let $A_1A_2A_3A_4A_5$ be a simple 5-point plane figure, and let d be any line in the plane of the figure. Let the common point of the line d and the side a_i opposite to A_i be denoted by B_i, and the common point of the lines A_iB_{i+1}, B_iA_{i+1} by C_{i+3}. Then the five lines A_iC_i have a point D in common.

Advanced Problem 4818, *American Mathematical Monthly*, **65**, (1958), 779.

4818. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let d_i be the sides of a complete quadrilateral, and A_{ij} be the vertex on d_i, d_j. Let t_i be the triangle formed by the sides other than d_i, and (O_i) denote the circumcircle of t_i. Denote the Simson line of a point S_i of (O_i) with respect to t_i by D_i.

Then prove that, if D_i and d_i are parallel for all i, (1) the line S_iO_p passes through the vertex A_{qr} $(i,\ p\neq q,\ r)$, and (2) the points S_i all lie on the Miquel circle (O).

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2 Solutions of Advanced Problem for *MONTHLY*

Solution to Problem 4102:

American Mathematical Monthly, **52**, (1945), 103-104.

SOLUTIONS

A Special Triangle Transversal

4102 [1943, 638]. *Proposed by Hüseyin Demir, Columbia University*

Let O and I be respectively the circumcenter and incenter of a given triangle ABC. Let A_0, B_0, C_0 be points taken respectively on BC, CA, AB so that the sums of the algebraic distances of each point to two other sides are equal to a given length l. Prove synthetically that: (1) The points A_0, B_0, C_0 are collinear; (2) The sum of distances to the sides of ABC of points on $A_0B_0C_0$ is the constant l; (3) the line $A_0B_0C_0$ is perpendicular to the line OI.

Solution by the Proposer. (1) The locus of points whose sum of distances to the sides CA, AB is l, is a straight line passing through A_0, and perpendicular to AI. Let B_c, C_b be points where this locus cuts CA, AB. Similarly we consider two other loci corresponding to B_0, C_0. Let $A'B'C'$ be the triangle formed by these three loci. We shall prove that the last triangle is in perspective with ABC, I being the center of perspective. This is obvious, because since A' is the intersection of two loci, its distances to CA, AB are equal, that is, A' belongs to AI. Similarly B', C' belong respectively to BI, CI. Thus applying Desargue's theorem we have collinearity of A_0, B_0, C_0.

(2) Let M be a point of $A_0B_0C_0$ with x, y, z its distances to BC, CA, AB. We shall prove that $x+y+z=l$. Consider the locus of points with $y+z=Cst$. This locus MQ (see figure) is parallel to B_cC_b, and $QQ_1=y+z$. Now, C_b, A_b having equal distances l to CA (see (1)) what we have to prove is that $QQ_2=x$. Draw MP parallel to BC, then $x=MX=PP_1$. Since $A'A_b$ is the bisector of $A_0A_bC_b$, we have $x=PP_1=PP_2$. It remains to prove that $QP\|C_bA_b$. This is true because the two triangles QMP, $C_bA_0A_b$ have two sides parallel, namely QM, C_bA_0 and MP, A_0A_b and they are in perspective, with C_0 as center of perspective. Therefore their third sides QP, C_bA_b must be parallel, that is $x=PP_2=QQ_2$.

(3) We shall prove two things: (a)—$A_0B_0C_0$ is the radical axis of circles (ABC) and $(A'B'C')$. (b)—The center O' of $(A'B'C')$ lies on OI, thus property (2) will be proved.

(a)—For, observe the relation $\overline{C_0A}\cdot\overline{C_0B}=\overline{C_0A'}\cdot\overline{C_0B'}$. This is true because the quadrilateral $ABB'A'$ is cyclic. (Note the equality of angles $A_bB'B=A'AB=\frac{1}{2}A$). Thus C_0 has equal powers with respect to the two circles. A similar property holds for A_0, B_0.

(b)—To prove that O' belongs to OI we shall remark that the locus of O' is a straight line when $A'B'C'$, whose sides are perpendicular to AI, BI, CI, varies, and since $A'B'C'$ is always in perspective with ABC, with I the center of perspective, O' will describe a straight line passing through I. It also passes through O. For, let A' be taken at the point where AI meets the circle (ABC). It is easy to see that B', C' will be similar points on the same circle. Thus O', the center of $(A'B'C')$, coincides with O, the center of (ABC). Therefore the radical axis $A_0B_0C_0$ of (ABC) and $(A'B'C')$ is perpendicular to the line OI passing through the centers O and O'.

Editorial Note. The first two theorems follow from similar triangles. The case of an isosceles ABC may be discarded. For, if say the sides AB, AC have equal lengths, then in consequence of symmetry about AI the same is true for AB_0, AC_0, B_0 and C_0 being respectively on AC and AB. It then follows that B_0C_0 is perpendicular to OI; the converse is true as well as parts (1) and (2), but A_0 has an exceptional position. The points A_0, B_0, C_0 are uniquely determined by the given constant l. The distances x_b, y_b, z_b for B_0 are such that $y_b=0$, $x_b+z_b=l$, *etc.* Let P be a point on the straight line of C_0B_0 and let it divide this segment in the ratio $\lambda:1$. Then we have

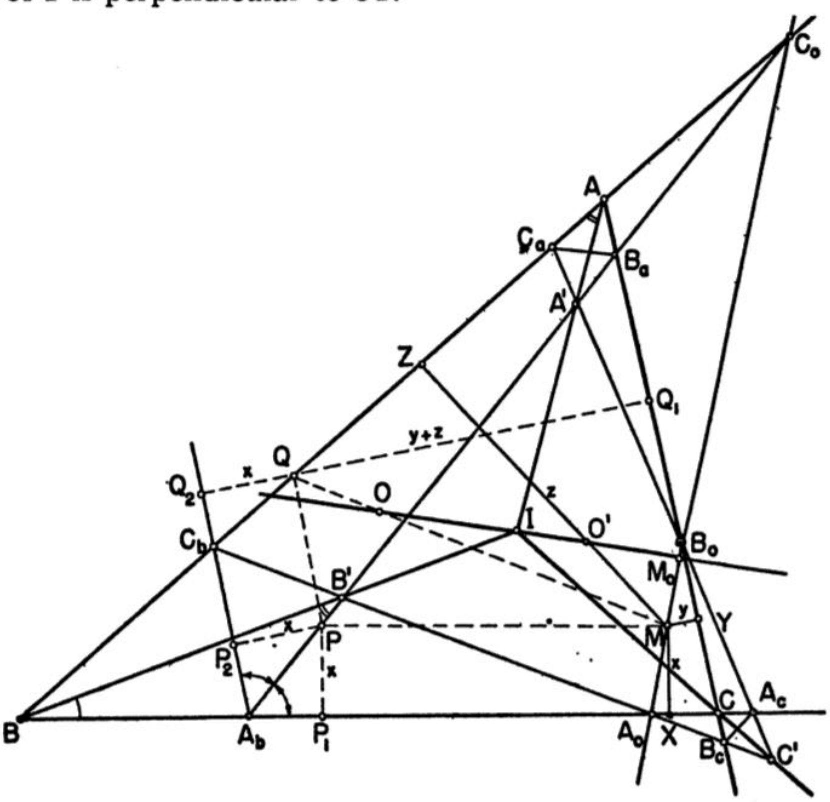
$$(\lambda+1)x = x_c + \lambda x_b, \qquad (\lambda+1)y = y_c, \qquad (\lambda+1)z = \lambda z_b,$$

where x, y, z are the distances for P. By addition we have

$$(\lambda+1)(x+y+z) = (x_c+y_c) + \lambda(x_b+z_b) = (\lambda+1)l;$$

and, if P is a finite point $x+y+z=l$. The straight line C_0B_0 meets BC in a finite point for which $x_a=0$ and $y_a+z_a=l$; hence this point is A_0. The two straight lines $A_0B_0C_0$ for different values of l are parallel; for, if they meet in a finite point, this point would have the sum of its distances equal to two different values. If P is a point not on B_0C_0 the line through it parallel to the latter meets the two sides in points different from B_0 and C_0. Hence the sum of its distances must be different from l; and this proves that the locus of points for a given l is the straight line $A_0B_0C_0$ for that value of l.

In the special case where $A'B'C'$ is inscribed in (O) the polar of C_0 passes through $(AA', BB')=I$, similarly, the polar of B_0 passes through I. Hence the polar B_0C_0 of I is perpendicular to OI.



Solution to Problem 4125:
American Mathematical Monthly, **52**, (1945), 523.

Trigonometric Determinants

4125 [1944, 352]. *Proposed by Hüseyin Demir, Columbia University*

Prove that

$$\begin{vmatrix} \sin\theta_1 & -e^{-i\theta_1} & 0 & 0 & \cdots 0 & 0 \\ \sin\theta_2 & e^{i\theta_2} & -e^{-i\theta_2} & 0 & \cdots 0 & 0 \\ \sin\theta_3 & 0 & e^{i\theta_2} & -e^{-i\theta_2} & \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sin\theta_n & 0 & 0 & 0 & \cdots 0 & e^{i\theta_n} \end{vmatrix} = \sin(\theta_1 + \theta_2 + \cdots + \theta_n).$$

Solution by Mary L. Boas, Tufts College. Put each $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$ and remove the factor $1/2i$ outside the determinant. Subtract from each element of the first column the sum of all the other elements in its row. The determinant then becomes

$$\frac{1}{2i}\begin{vmatrix} e^{i\theta_1} & -e^{-i\theta_1} & 0 & \cdots 0 & 0 \\ 0 & e^{i\theta_2} & -e^{-i\theta_2} & \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots e^{i\theta_{n-1}} & -e^{-i\theta_{n-1}} \\ -e^{-i\theta_n} & 0 & 0 & \cdots 0 & e^{i\theta_n} \end{vmatrix}.$$

Expand by elements of the first column. The minor of $e^{i\theta_1}$ is $e^{i(\theta_2+\theta_3+\cdots+\theta_n)}$ since all elements below the main diagonal of this minor are zero. The minor of $-e^{-i\theta_n}$ is $(-1)^{n-1}e^{-i(\theta_1+\theta_2+\cdots+\theta_{n-1})}$ since all elements above its main diagonal are zero. Therefore the determinant equals

$$\frac{1}{2i}\left[e^{i(\theta_1+\theta_2+\cdots+\theta_n)} - e^{-i(\theta_1+\theta_2+\cdots+\theta_n)}\right] = \sin(\theta_1 + \theta_2 + \cdots + \theta_n).$$

Solved also by E. F. Allen, Murray Barbour, C. B. Barker, Jr., Shepard Bartnoff, R. P. Boas, Jr., Mrs. R. C. Buck, Howard Eves, Clifford Gardner, P. C. Hammer, R. Hamming, J. F. Hofmann, L. M. Kelly, E. Lukacs, Norman Miller, Henry Nelson, Ivan Niven, H. N. Shapiro, Robert Steinberg, R. H. Wilson, Jr., and the proposer.

Editorial Note. About half of the solutions used induction proofs and about the same number used simple determinant transformations without induction. Hammer considered the transformation of the determinant and its value by replacing θ_j by $\pi/2-\theta_j$ which gives after reduction a determinant with $a_{j1}=\cos\theta_j$ in the first column and the principal diagonal $\cos\theta_1$, $e^{-i\theta_2}$, $e^{-i\theta_3}$, $\cdots$ and the parallel above it $e^{i\theta_1}$, $e^{i\theta_2}$, $e^{i\theta_3}$, $\cdots$ with zeros in the remaining places. He found for the value of the determinant $\cos\sum\theta_j$ if n is odd, and $-i\sin\sum\theta_j$ if n is even. A simpler procedure is to make the same change in the first column but to alter the principal diagonal to $\cos\theta_1$, $-e^{i\theta_2}$, $-e^{i\theta_3}$, $\cdots$, $-e^{i\theta_n}$ and leave the rest of the original determinant unaltered. The value of this determinant is the same as that for Hammer's determinant.

Solution to Problem 4134:
American Mathematical Monthly, **52**, (1945), 587.

Angle Trisection

4134 [1944, 475]. ***Proposed by Hüseyin Demir, Columbia University***

Let $C_1^1C_2^1C_3^1$ be the inscribed triangle of a reference triangle $A_1A_2A_3$, and $C_1^2C_2^2C_3^2$ be that of $C_1^1C_2^1C_3^1$, and so on, obtaining a triangle $C_1^nC_2^nC_3^n$ after n steps. Denoting the angles of the nth triangle by C_i^n, prove that

1. $(C_i^n-\pi/3)(A_i-\pi/3)=(-1)^n2^{-n}$.

2. The limit of the direction of $C_2^nC_3^n$ as $n\to\infty$, is the direction of one of the trisectrices of the angle $(A_2A_3,\ C_2^1C_3^1)$, and from that observe a method of trisecting an angle by ruler and compass in infinitely many steps.

Solution by Howard Eves, College of Puget Sound. 1. Designating the incenter of $A_1A_2A_3$ by I we have $C_2^1IC_3^1=2C_1^1$. Therefore $A_1+2C_1^1=\pi$. Similarly, $A_i+2C_i^1=\pi$, or

$$(C_i^1-\pi/3)/(A_i-\pi/3)=-2^{-1}.$$

By the same process

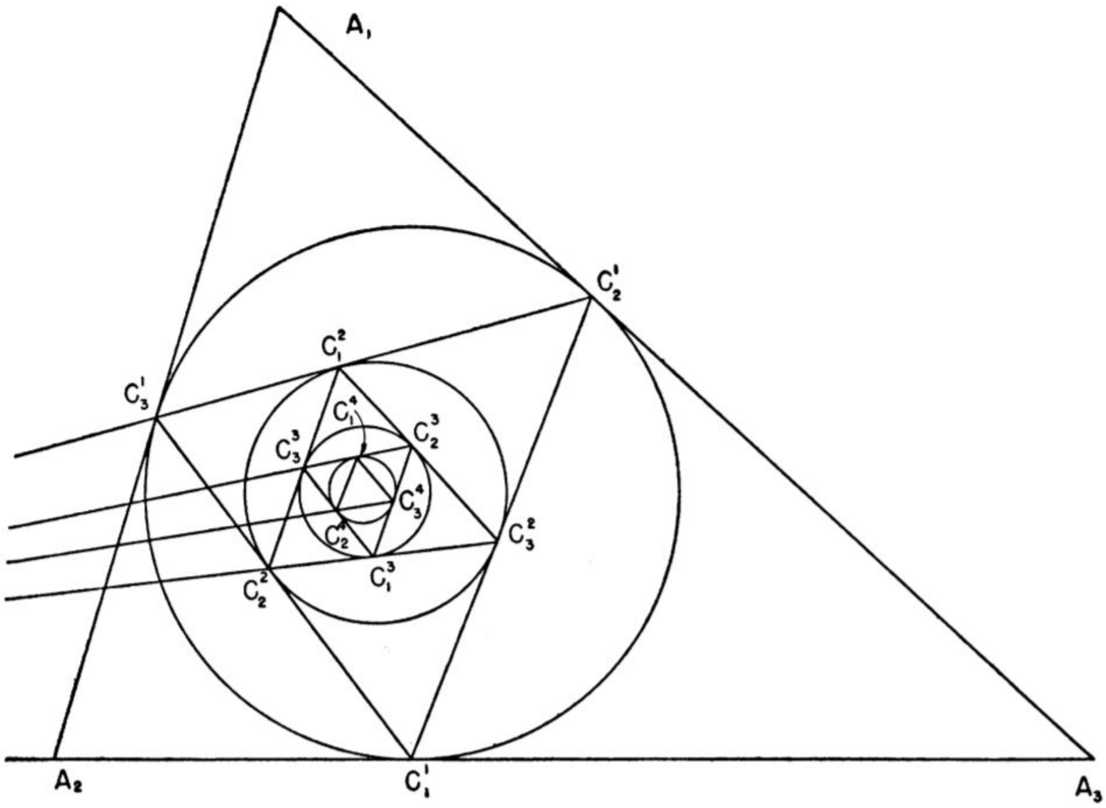
$$(C_i^j-\pi/3)/(C_i^{j-1}-\pi/3)=-2^{-1}, \qquad j=2,\cdots,n.$$

By multiplication we then get

$$(C_i^n-\pi/3)/(A_i-\pi/3)=(-1)^n2^{-n}.$$

2. Now

$$\begin{aligned}\lim_{n\to\infty}\left[\text{angle }(A_2A_3,\, C_2^nC_3^n)\right] &= \sum_{n=1}^{\infty}\text{angle }(C_1^{2n-2}C_2^{2n-2},\, C_1^{2n}C_2^{2n}), \quad \text{where} \quad C_i^0 \equiv A_i \\ &= \sum_1^{\infty}(C_3^{2n-1} - C_1^{2n}) \\ &= \sum_1^{\infty}\left[(C_3^{2n-1} - \pi/3) - (C_1^{2n} - \pi/3)\right] \\ &= \sum_1^{\infty}\left[-\,2^{-2n+1}(A_3 - \pi/3) - 2^{-2n}(A_1 - \pi/3)\right],\end{aligned}$$



by the relations of part 1 above. The infinite sum on the right reduces to

$$\begin{aligned}\sum_1^{\infty}[\pi - A_1 - 2A_3]2^{-2n} = (A_2 - A_3)\sum_1^{\infty}2^{-2n} = \tfrac{1}{3}(A_2 - A_3) &= \tfrac{2}{3}\left(\frac{A_2 - A_3}{2}\right); \\ &= (\tfrac{2}{3})\left[\tfrac{1}{2}(A_2 - \pi/3) - \tfrac{1}{2}(A_3 - \pi/3)\right] \\ &= (\tfrac{2}{3})\left[-\,(C_2^1 - \pi/3) + (C_3^1 - \pi/3)\right] \\ &= (\tfrac{2}{3})(C_3^1 - C_2^1) = (\tfrac{2}{3})\text{ angle }(A_2A_3,\, C_2^1C_3^1).\end{aligned}$$

This establishes part 2. The method suggested here for asymptotically obtaining one of the angle trisectors of a given acute angle $(A_2A_3,\ C_2^1C_3^1)$ is apparent. It is needless to say, however, that there are many better euclidean asymptotic constructions for trisecting an angle.

Solved also by the proposer.

Editorial Note. Below are some references to this MONTHLY regarding approximate methods of angle trisection with limits for the error:

1932, 478, *Angle Division*, article by E. C. Kennedy; 2972 [1925, 95]; 3114 [1925, 483]; 3522 [1933, 303]; 3563 [1934, 113]; The method of Pappus using conics 3490 [1932, 243].

Solution to Problem 4193:
American Mathematical Monthly, **54**, (1947), 349.

Concurrent Lines in a Pentagon

4193 [1946, 160]. *Proposed by Hüseyin Demir, Columbia University*

If on the sides of an arbitrary pentagon $A_1A_2A_3A_4A_5$ the triangles $B_iA_{i+2}A_{i+3}$ (with indices reduced mod 5) are constructed such that $B_iA_{i+2}\|A_iA_{i+1}$, and $B_iA_{i+3}\|A_iA_{i+4}$, then the lines A_iB_i concur in a point C.

Solution by J. W. Clawson, Ursinus College, Collegeville, Pennsylvania. Take the triangle $A_1A_2A_4$ as the triangle of reference for a system of homogeneous trilinear coördinates. Let A_1 be $(1, 0, 0)$, A_2 $(0, 1, 0,)$, A_3 (d, e, f), A_4 $(0, 0, 1)$, A_5 (k, l, m).

Then the equations of the line through A_3 parallel to A_1A_2 and of the line through A_4 parallel to A_1A_5 are

$$afx + bfy - (ad + be)z = 0, \qquad alx + (bl + cm)y = 0.$$

Thus B_1 has the coördinates

$$(bl + cm)(ad + be), \qquad -al(ad + be), \qquad acfm;$$

and the equation of A_1B_1 is

$$cfmy + l(ad + be)z = 0.$$

In the same way the equations of A_2B_2 and A_4B_4 are found to be

$$cfmx + d(ak + bl)z = 0, \qquad l(ad + be)x - d(ak + bl)y = 0.$$

These three lines are easily seen to be concurrent in a point C which has the coördinates

$$d(ak + bl), \qquad l(ad + be), \qquad -cfm.$$

Using the triangles $A_1A_3A_4$ and $A_2A_4A_5$ we can prove in the same way that A_3B_3 and A_5B_5 also pass through the point C.

Solved also by the Proposer.

Editorial Note. Clawson gave a second proof using the converse of Ceva's Theorem. The Proposer employed the pencils of lines A_3B_3, A_4B_4 formed when the side A_2A_3 rotates about A_3, other sides remaining fixed; since the correspondence between A_3B_3 and A_4B_4 is homographic, the locus of the intersection C of the rays A_3B_3, A_4B_4 is a conic; this conic decomposes into A_1B_1 and A_3A_4, thus giving the proof.

Solution to Problem 4215:
American Mathematical Monthly, **55**, (1948), 34.

Hermite Polynomials

4215 [1946, 470]. *Proposed by Hüseyin Demir, Columbia University*

Prove that the Hermite polynomials defined as follows

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$

have the property

$$n! \sum_{p=0}^{n} \frac{H_p^2(x)}{p!} = H_{n+1}^2(x) - H_n(x) H_{n+2}(x).$$

Solution by Hsien-yü Hsü Yenching University, Peiping, China. In Polya-Szegö, *Aufgaben und Lehrsätze* II, pp. 294–295, Hermite polynomials are defined as follows

$$h_n(x) = \frac{1}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2},$$

and satisfy the difference equation

$$nh_n(x) = - xh_{n-1}(x) - h_{n-2}(x), \qquad n = 2, 3, \cdots .$$

We notice that

$$H_n \equiv H_n(x) = (-1)^n n! h_n(x),$$

whence the difference equation is

$$H_n = xH_{n-1} - (n-1)H_{n-2}, \qquad n = 2, 3, \cdots . \tag{1}$$

Upon eliminating x from this equation and the analogous equations for H_{n+1} and H_{n+2} we obtain immediately

$$\begin{aligned} H_{n+1}^2 - H_nH_{n+2} &= n(H_n^2 - H_{n+1}H_{n-1}) + H_n^2 \\ &= n(n-1)(H_{n-1}^2 - H_nH_{n-2}) + nH_{n-1}^2 + H_n^2 \\ &= \cdots \\ &= n!\left\{(H_1^2 - H_2H_0) + \frac{1}{1!}H_1^2 + \frac{1}{2!}H_2^2 + \cdots + \frac{1}{n!}H_n^2\right\} \\ &= n! \sum_{p=0}^{n} H_p^2/p!, \end{aligned}$$

since $H_1^2 - H_2H_0 = 1$.

Solved also by F. E. Cothran, A. B. Farnell, L. M. Kelly, Norman Miller, S. T. Parker, W. A. Pierce, W. H. Spragens, M. S. Webster, M. Wyman, Professor Otto Szász's class, and the Proposer.

Editorial Note. Several solvers mentioned that equation (1) of the above solution is found in Dunham Jackson, *Fourier Series and Orthogonal Polynomials*, p. 176, ff. The solution by members of Professor Szász's class in Orthogonal Developments proceeds from the (so-called) Christoffel's formula

$$\sum_{p=0}^{n} \frac{H_p(x)H_p(y)}{p!} = \frac{H_{n+1}(x)H_n(y) - H_n(x)H_{n+1}(y)}{n!(x-y)} .$$

See Szegö, *Orthogonal Polynomials*, p. 102. Webster's solution employs the relation

$$H_m(x)H_n(x) = n!\sum_{r=0}^{n} \binom{m}{n-r} \frac{H_{m-n+2r}}{r!} \qquad (m \geqq n),$$

established by E. Feldheim, *Journal of the London Mathematical Society*, 1938, pp. 22–29.

Solution to Problem 4679:
American Mathematical Monthly, **63**, (1956), 191.

Ten Concyclic Orthopoles

4679 [1956, 191]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

If $A_1A_2A_3A_4A_5$ is a cyclic pentagon and if Ω_{ij} denotes the orthopole of the line A_iA_j with respect to the triangle formed by the remaining three vertices, then prove that the ten points Ω_{ij} all lie on a circle.

Solution by Chih-yi Wang, University of Minnesota. We make use of the following known

THEOREM. *If a line meets the circumcircle of a triangle, the Simson lines of the points of intersection with the circle meet in the orthopole of the line for the triangle.* (See Court, *College Geometry*, 2nd ed., p. 289.)

Let the circumcircle $A_1A_2A_3A_4A_5$ be the unit circle, and the coordinates of A_i be $(\cos\theta_i, \sin\theta_i)$, $i=1, 2, \cdots, 5$. For definiteness let us find the coordinates of Ω_{12}. The equations of the Simson lines of A_1 and of A_2 are given by

$$\begin{aligned} y - \tfrac{1}{2}(\sin\theta_j + \sin\theta_3 + \sin\theta_4) &= \tfrac{1}{2}\sin(\theta_3 + \theta_4 - \theta_j) \\ &= \tan\tfrac{1}{2}(\theta_3 + \theta_4 + \theta_5 - \theta_j)\left[x - \tfrac{1}{2}(\cos\theta_3 + \cos\theta_4 + \cos\theta_j) + \tfrac{1}{2}\cos(\theta_3 + \theta_4 - \theta_j)\right], \end{aligned}$$

for $j=1, 2$. By solving the simultaneous equations we obtain

$$\Omega_{12} = \left(\alpha + \tfrac{1}{2}\cos(\theta_3 + \theta_4 + \theta_5 - \theta_1 - \theta_2),\ \beta + \tfrac{1}{2}\sin(\theta_3 + \theta_4 + \theta_5 - \theta_1 - \theta_2)\right),$$

where $\alpha = \frac{1}{2}\sum\cos\theta_k$, $\beta = \frac{1}{2}\sum\sin\theta_k$, $k=1, 2, \cdots, 5$. Since α and β are symmetric functions, by interchanging the subscripts we see readily that the ten points Ω_{ij} all lie on the circle of radius $\frac{1}{2}$ with center (α, β).

Also solved by J. W. Clawson, R. Goormaghtigh, O. J. Ramler, Sister M. Stephanie, and the proposer.

Editorial Note. Goormaghtigh gave this theorem in *Mathesis*, 1939, p. 312. Ramler gives an extension to the cyclic heptagon. If Ω_{ijk} denotes the Kantor point of a triangle $A_iA_jA_k$ with respect to the quadrangle formed by the remaining four vertices, then the thirty-five points Ω_{ijk} all lie on a circle one-half as large as the circumcircle of the heptagon.

Solution to Problem 4695:
American Mathematical Monthly, **64**, (1957), 437.

Simson Line and Euler Line

4695 [1956, 426]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that if in a cyclic quadrangle the Simson line of one vertex with respect to the triangle formed by the other three is perpendicular to the Euler line of that triangle, then the same property holds for the other vertices of the quadrangle.

Solution by Sister M. Stephanie, Georgian Court College, Lakewood, N. J. Using complex coordinates and taking the circle to be the unit circle, let the vertices of the quadrangle be t_1, t_2, t_3, t_4, $|t_i| = 1$. The Simson line of t_1, with respect to the triangle formed by t_2, t_3, t_4, has the equation

$$2t_1^2 z - 2t_1 s_3 \bar{z} + s_3 + t_1 s_2 - t_1^3 - t_1^2 s_1 = 0,$$

where s_1, s_2 and s_3 are the elementary symmetric functions of t_2, t_3 and t_4. The Euler line of triangle t_2, t_3, t_4 has the equation $s_2 z - s_1 s_3 \bar{z} = 0$. If the two lines are perpendicular, one clinant is the negative of the other, whence $s_3/t_1 = -s_1 s_3/s_2$, or upon simplifying, $t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = 0$. The symmetry of this result guarantees that the property holds equally for any vertex.

Also solved by J. W. Clawson, G. W. Courter, R. Deaux, Beckham Martin, O. J. Ramler, Robert Sibson, Chih-yi Wang, and the proposer.

Solution to Problem 4710:
American Mathematical Monthly, **64**, (1957), 601.

Isotomic Lines

4710 [1956, 669]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that if in a complete quadrangle inscribed in a circle (O) one pair of opposite sides are isotomic lines with respect to a triangle inscribed in (O), then the remaining pairs of opposite sides are also isotomic lines with respect to the same triangle.

I. *Solution by O. J. Ramler, Catholic University of America.* Using a system of conjugate coordinates we take the circle (O) as the unit circle, and on it points whose vector coordinates are T_1, T_2, T_3 as the inscribed triangle and t_1, t_2, t_3, t_4 as the inscribed complete quadrangle. Then the line t_2t_3 intersects the side T_2T_3 of the triangle in a point whose vector coordinate is

$$z = \frac{T_2T_3(t_2 + t_3) - (T_2 + T_3)t_2t_3}{T_2T_3 - t_2t_3}.$$

Similarly t_1t_4 meets side T_2T_3 where

$$z' = \frac{T_2T_3(t_4 + t_1) - (T_2 + T_3)t_1t_4}{T_2T_3 - t_1t_4}.$$

The hypothesis implies $T_3 - z = z' - T_2$ which becomes, upon making proper substitutions and simplifying

$$T_2^2T_3^2(T_2 + T_3 - s_1) + T_2T_3s_3 - (T_2 + T_3)s_4 = 0,$$

where s_1, s_3, s_4 are elementary symmetric functions of t_1, t_2, t_3, t_4. The symmetry of this result establishes the proposed theorem.

II. *Solution by Roland Deaux, Faculté Polytechnique, Mons, Belgium.* Circle (O) may be replaced by any conic Γ. Let ABC and $PQRS$ be the triangle and the quadrangle inscribed in Γ. By virtue of Desargues' theorem, the four conics Γ, (PQ, RS), (PR, QS), (PS, QR) determine on each side of ABC four pairs of an involution. This involution, defined by Γ and the isotomic lines PQ, RS, has for double points the midpoint and the point at infinity of the side. Hence the property.

Also solved by J. W. Clawson, N. A. Court, R. Goormaghtigh, Josef Langr, and the proposer.

Solution to Problem 4735:
American Mathematical Monthly, **65**, (1958), 128.

Concurrent Lines

4735 [1957, 277]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let $A_1A_2A_3A_4A_5$ be a simple 5-point plane figure, and let d be any line in the plane of the figure. Let the common point of the line d and the side a_i opposite to A_i be denoted by B_i, and the common point of the lines A_iB_{i+1}, B_iA_{i+1} by C_{i+3}. Then the five lines A_iC_i have a point D in common.

Solution by E. J. F. Primrose, The University, Leicester, England. There is a unique polarity P for which each A_i is the pole of the opposite side a_i (Coxeter, *The Real Projective Plane*, 5.65). We consider the 4-point $A_1C_1B_3B_4$. The pole of A_1B_3 for P is A_3, so A_1B_3 and C_1B_4 are conjugate lines, and similarly A_1B_4 and C_1B_3 are conjugate lines. By the dual of Hesse's theorem (Coxeter, 5.55), A_1C_1 and B_3B_4 are conjugate lines, so A_1C_1 passes through D, the pole of d for P. By a similar argument, all the lines A_iC_i pass through D.

Also solved by W. B. Carver, J. W. Clawson, R. Deaux, and the proposer.

Solution to Problem 4818:
American Mathematical Monthly, **66**, (1959), 732.

A Property of the Miquel Circle of a Complete Quadrilateral

4818 [1958, 779]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let d_i be the sides of a complete quadrilateral, and A_{ij} be the vertex on d_i, d_j. Let t_i be the triangle formed by the sides other than d_i, and (O_i) denote the circumcircle of t_i. Denote the Simson line of a point S_i of (O_i) with respect to t_i by D_i.

Then prove that, if D_i and d_i are parallel for all i, (1) the line S_iO_p passes through the vertex A_{qr} $(i,\ p \neq q,\ r)$, and (2) the points S_i all lie on the Miquel circle (O).

Solution by the proposer. (1) Let the projections of S_i and O_p on d_r be denoted by U_{ir}, V_{pr}. Then, using directed angles, we have

$$\begin{aligned}
\measuredangle V_{pr}O_pA_{qr} &= \measuredangle A_{ir}A_{qi}A_{qr} && \text{(from the circle } (O_p)),\\
&= \measuredangle U_{ir}U_{iq}A_{qr} && \text{(since } D_i \text{ is parallel to } d_i),\\
&= \measuredangle U_{ir}S_iA_{qr}. && \text{(from the circle } S_iU_{ir}A_{qr}U_{iq}).
\end{aligned}$$

Now, since O_pV_{pr} is parallel to S_iU_{ir}, we get the required collinearity of S_i, O_p, A_{qr}.

(2) Since the line of centers O_pO_q is perpendicular to the radical axis FA_{pq}, where F is the Miquel point, we have successively

$$\begin{aligned}
\measuredangle\, O_pO_rO_q &= \measuredangle\, A_{qi}FA_{pi}, \\
&= \measuredangle\, A_{qr}A_{pq}A_{pi} && \text{(from the circle } (O_r)), \\
&= \measuredangle\, A_{qr}A_{pq}A_{pr}, \\
&= \measuredangle\, A_{qr}S_iA_{pr} && \text{(from the circle } (O_i)), \\
&= \measuredangle\, O_pS_iO_q, && \text{(from property (1))},
\end{aligned}$$

and S_i lies on the Miquel circle (O).

Also solved by A. E. Landry.

3 Elementary Problems for *MONTHLY*

List of Elementary Problems:

Elementary Problem 1134, *American Mathematical Monthly*, **61**, (1954), 568.

E 1134. ***Proposed by Hüseyin Demir, Zonguldak, Turkey***

Prove that a square integer is not a perfect number.

Elementary Problem 1160, *American Mathematical Monthly*, **62**, (1955), 182.

E 1160. ***Proposed by Hüseyin Demir, Zonguldak, Turkey***

Prove that in a complete quadrilateral the isotomic line of any side with respect to the triangle formed by the other three is parallel to the Newton line of the quadrilateral.

Elementary Problem 1197, *American Mathematical Monthly*, **63**, (1956), 39.

E 1197. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let ABC be a right triangle and CH the altitude on the hypotenuse AB. Show that the sum of the radii of the inscribed circles of triangles ABC, HCA, HCB is equal to CH.

Elementary Problem 1209, *American Mathematical Monthly*, **63**, (1956), 186.

E 1209. ***Proposed by Hüseyin Demir, Zonguldak, Turkey***

Let ABC be any triangle and (I) its incircle. Let (I) touch BC, CA, AB at D, E, F, and intersect the cevians BE, CF at E', F' respectively. Show that the anharmonic ratio $D(E, F, E', F')$ is the same for all triangles ABC.

Elementary Problem 1217, *American Mathematical Monthly*, **63**, (1956), 342.

E 1217. ***Proposed by Hüseyin Demir, Zonguldak, Turkey***

Evaluate

$$\prod_{d|n} d^{\phi(n/d)+\phi(d)}.$$

Elementary Problem 1778, *American Mathematical Monthly*, **72**, (1965), 420.

E 1778. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

If R, r, r_1, r_2, r_3 are the circumradius, inradius and exradii of a triangle, prove that

$$\frac{1}{r^3}-\frac{1}{r_1^3}-\frac{1}{r_2^3}-\frac{1}{r_3^3}=\frac{12R}{r\cdot r_1\cdot r_2\cdot r_3}\,.$$

Elementary Problem 1779, *American Mathematical Monthly*, **72**, (1965), 420.

E 1779. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

If h_i and r_i are the altitudes and exradii of a triangle prove that

$$\frac{r_1}{h_1}+\frac{r_2}{h_2}+\frac{r_3}{h_3}\geqq 3.$$

Elementary Problem 1877, *American Mathematical Monthly*, **73**, (1966), 410.

E 1877. *Proposed by Huseyin Demir, Middle East Technical University, Ankara*

Let $ABCDE$ be a convex pentagon inscribed in a unit circle with AE as diameter, and let $AB=a$, $BC=b$, $CD=c$, $DE=d$. Then prove that

$$a^2+b^2+c^2+d^2+abc+bcd<4.$$

Elementary Problem 1878, *American Mathematical Monthly*, **73**, (1966), 410.

E 1878. *Proposed by Huseyin Demir, Middle East Technical University, Ankara*

Let $A_1A_2\cdots A_n$ be a regular polygon inscribed in circle (O) of radius R. Denote the incenter of the triangle $A_{i-1}A_iA_{i+1}$ (indices mod n) by I_i, and that of the triangle formed by A_iA_{i+2}, $A_{i+2}A_{i+1}$, $A_{i+1}A_{i+3}$ by J_i. Then show that the points $I_1,\cdots,I_n$ and $J_1,\cdots,J_n$ all lie on the same circle of radius $R'=R\cos(3\pi/2n)/\cos(\pi/2n)$.

Elementary Problem 2100, *American Mathematical Monthly*, **75**, (1968), 670.

E 2100. *Proposed by H. Demir, Middle East Technological University, Ankara, Turkey*

Show that any five of the relations

$$(1)\ \frac{x-a_1}{a_1-a_2}=\frac{a-b}{b-c},\qquad (2)\ \frac{x-b_1}{b_1-b_2}=\frac{b-c}{c-a},\qquad (3)\ \frac{x-c_1}{c_1-c_2}=\frac{c-a}{a-b},$$

$$(4)\ x+a=b_2+c_1,\qquad (5)\ x+b=c_2+a_1,\qquad (6)\ x+c=a_2+b_1$$

imply the sixth. Interpret this set of consistent relations geometrically letting a, b, c be the affixes, in the complex plane, of a triangle of reference ABC and other numbers be those of other points.

Elementary Problem 2101, *American Mathematical Monthly*, **75**, (1968), 670.

E 2101. *Proposed by H. Demir, Middle East Technological University, Ankara, Turkey*

ABC is a triangle. Let P_a denote the parabola tangent to the sides AB, AC at B, C respectively. The parabolas P_b and P_c are similarly defined. Let these parabolas intersect to the points A', B', C' inside ABC. Denote the areas of the (curvilinear) triangular regions ABC, $A'B'C'$, $AB'C'$, $BC'A'$, $CA'B'$, $A'BC$, $B'CA$, $C'AB$, by $\triangle$, $\triangle_0$, $\triangle_a'$, $\triangle_b'$, $\triangle_c'$, $\triangle_a''$, $\triangle_b''$, $\triangle_c''$. Then prove

(1) $\triangle_a'=\triangle_b'=\triangle_c'=(\triangle_1)$, $\triangle_a''=\triangle_b''=\triangle_c''=(\triangle_2)$,
(2) $\triangle_0:\triangle_1:\triangle_2:\triangle=15:17:5:81$.

Elementary Problem 2109, *American Mathematical Monthly*, **75**, (1968), 780.

E 2109. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

Let ABC be a triangle and A' be any fixed point on the side BC. Construct the inscribed triangle $A'B'C'$ which is directly similar to a given triangle XYZ.

Elementary Problem 2110, *American Mathematical Monthly*, **75**, (1968), 780.

E 2110. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

If, in a plane, the triangles AUV, VBU, UVC are directly similar to a given triangle, then so is ABC.

Elementary Problem 2124, *American Mathematical Monthly*, **75**, (1968), 899.

E 2124. ***Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey***

Construct on the sides BC, CA, AB of a triangle ABC, exteriorly, the squares $BCDE$, $ACFG$, $BAHK$ and build the parallelograms $FCDQ$, $EBKP$. Show that APQ is an isosceles right triangle.

Elementary Problem 2160, *American Mathematical Monthly*, **76**, (1969), 300.

E 2160. ***Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey***

Let p_i, x_i be the distances of an interior or a boundary point P of a triangle $A_1A_2A_3$ from the vertex A_i and the side opposite to A_i, $i=1, 2, 3$, with r the inradius. Prove the inequalities

(a)
$$\sum_{i=1}^{3} p_i(\tfrac{1}{2}\sin A_i) \leqq \sum_{i=1}^{3} x_i \leqq \sum_{i=1}^{3} p_i \sin(\tfrac{1}{2}A_i).$$

(b)
$$p_2p_3 + p_3p_1 + p_1p_2 \geqq 8x_1x_2x_3/r.$$

Elementary Problem 2213, *American Mathematical Monthly*, **77**, (1970), 79.

E 2213. *Proposed by H. Demir, Middle East Technical University, Ankara, Turkey*

Let us say that a (planar) polygon has the *Nagel property* if the lines through the vertices of the polygon and bisecting the perimeter of the polygon are concurrent. It is known that all triangles have the Nagel Property and that not all quadrilaterals have the property. Determine the simple nondegenerate quadrilaterals that have the Nagel property.

Elementary Problem 2311, *American Mathematical Monthly*, **78**, (1971), 793.

E 2311. ***Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey***

Prove that, if a quadrilateral $A_1A_2A_3A_4$ can be inscribed in a circle, then the (six) lines drawn from the midpoints of A_pA_q perpendicular to A_rA_s (p, q, r, s are distinct) are concurrent.

Elementary Problem 2312, *American Mathematical Monthly*, **78**, (1971), 793.

E 2312. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey*

Let D be a point in the plane of a positively oriented triangle ABC and let AD, BD, CD intersect the respective opposite sides in A_1, B_1, C_1. If the oriented segments $\overline{BA}_1$, $\overline{CB}_1$, $\overline{AC}_1$ are equal $(=\delta)$, then D is uniquely determined and lies in the interior of ABC. (Notice the analogy between D and the Brocard point Ω.)

Elementary Problem 2363, *American Mathematical Monthly*, **79**, (1972), 662.

E 2363. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

Characterize pairs of spherical triangles ABC and $A'B'C'$ for which $A' = a$, $B' = b$, $C' = c$, $A = a'$, $B = b'$, $C = c'$.

Elementary Problem 2462, *American Mathematical Monthly*, **81**, (1974), 281.

E 2462. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey*

Let P be a point interior to the triangle $A_1A_2A_3$. Denote by R_i the distance from P to the vertex A_i, and denote by r_i the distance from P to the side a_i opposite to A_i. The Erdös-Mordell inequality asserts that

$$R_1 + R_2 + R_3 \geqq 2(r_1 + r_2 + r_3).$$

Prove that the above inequality holds for every point P in the plane of $A_1A_2A_3$ when we make the interpretation $R_i \geqq 0$ always and r_i is positive or negative depending on whether P and A_i are on the same side of a_i or on opposite sides.

Elementary Problem 2625, *American Mathematical Monthly*, **83**, (1976), 812.

E 2625. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

Let A_i $(i = 0, 1, 2, 3 \pmod 4)$ be four points on a circle Γ. Let t_i be the tangent to Γ at A_i and let p_i and q_i be the lines parallel to t_i passing through the points A_{i-1} and A_{i+1}, respectively. If $B_i = t_i \cap t_{i+1}$, $C_i = p_i \cap q_{i+1}$, show that the four lines B_iC_i have a common point.

Elementary Problem 3135, *American Mathematical Monthly*, **93**, (1986), 215.

E 3135. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

For a scalene triangle ABC inscribed in a circle, prove that there is a point D on the arc of the circle opposite to some vertex whose distance from this vertex is the sum of its distances from the other two vertices.

Show how D may be constructed with straightedge and compass.

Elementary Problem 3164, *American Mathematical Monthly*, **93**, (1986), 566.

E 3164. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let s, t be the lengths of the tangent line segments to an ellipse from an exterior point. Find the extreme values of the ratio s/t.

Elementary Problem 3422, *American Mathematical Monthly*, **98**, (1991), 158.

E 3422. *Proposed by H. Demir and C. Tezer, Middle East Technical University, Ankara, Turkey.*

Suppose F and F' are points situated symmetrically with respect to the center of a given circle, and suppose S is a point on the circle not on the line FF'. Let P and P' be the second points of intersection of SF and SF' respectively with the circle. If the tangents to the circle at P and P' intersect at T, prove that the perpendicular bisector of FF' passes through the midpoint of the line segment ST.

Elementary Problem 3469, *American Mathematical Monthly*, **98**, (1991), 955.

E 3469. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

Suppose P is a point in the interior of triangle ABC and suppose AP, BP, CP meet the lines BC, CA, AB respectively at the points D, E, F. Prove that the centroids of the six triangles PBD, PDC, PCE, PEA, PAF, PFB lie on a conic if and only if P lies on at least one of the three medians of the triangle.

4 Solutions of Elementary Problem for *MONTHLY*

Solution to Problem 1134:
American Mathematical Monthly, **62**, (1955), 257.

Squares and Perfect Numbers

E 1134 [1954, 568]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that a square integer is not a perfect number.

I. *Solution by C. F. Pinzka, Princeton, N. J.* If $N=\Pi p^{2a}$, the sum of the divisors of N is

$$\Pi \frac{p^{2a+1}-1}{p-1}.$$

Since the latter is always odd, it cannot equal $2N$ as required for a perfect number.

II. *Solution by C. D. Olds, San Jose State College.* Euler proved that an odd perfect number must have the form $r^{4k+1}P^2$, where r is a prime of the form $4n+1$. An even perfect number must be of Euclid's type, that is, of the form $2^{p-1}(2^p-1)$ where 2^p-1 is a prime. Thus a square cannot be perfect.

Solution to Problem 1160:
American Mathematical Monthly, **62**, (1955), 658.

A Property of the Newton Line of a Complete Quadrilateral

E 1160 [1955, 182]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that in a complete quadrilateral the isotomic line of any side with respect to the triangle formed by the other three is parallel to the Newton line of the quadrilateral.

I. *Solution by the Proposer.* Let d be one of the four sides of the quadrilateral and let ABC be the corresponding triangle. Denote the intersections of d with the sides BC, CA, AB of triangle ABC by α, β, γ. The isotomic line IJK of $\alpha\beta\gamma$ with respect to triangle ABC is obtained by taking the symmetrics I, J, K of the points α, β, γ with respect to the midpoints A', B', C' of the sides BC, CA, AB of triangle ABC. Let the midpoints of $A\alpha$, $B\beta$, $C\gamma$ be denoted by I', J', K'. These points of the Newton line of the quadrilateral are evidently on the sides of the medial triangle $A'B'C'$ of triangle ABC. It is easy to see that the complete quadrilateral formed by triangle ABC and line IJK is similar to that formed by triangle $A'B'C'$ and line $I'J'K'$, for, firstly, triangles ABC and $A'B'C'$ are similar, and are in the ratio 2:1, and secondly,

$$BI = C\alpha = 2(B'I'), \qquad AJ = C\beta = 2(A'J'), \qquad AK = B\gamma = 2(A'K').$$

This proves that the lines IJK and $I'J'K'$ are parallel.

II. *Solution by Sister M. Stephanie, Georgian Court College, Lakewood, N.J.* Since there is one and only one parabola tangent to four lines, let us consider the complete quadrilateral as tangent to the parabola (referred to rectangular coordinates) $y^2=4ax$. Then $y=m_ix+a/m_i$, $i=1, 2, 3, 4$, may be taken as the equations of the four sides 1, 2, 3, 4 of the quadrilateral. Point

$$(a/m_1m_2,\ a/m_1 + a/m_2)$$

is the intersection of sides 1 and 2; other intersections are similarly given. The midpoint of the side 2 of triangle 123 has coordinates

$$(a/2)\left[(m_1 + m_3)/m_1m_2m_2,\ 2/m_2 + 1/m_1 + 1/m_3\right].$$

If (x, y) is the point on side 2 isotomic to the intersection of side 4 with side 2, then

$$y + a/m_2 + a/m_4 = 2a/m_2 + a/m_1 + a/m_3,$$

whence

$$y = a(1/m_1 + 1/m_2 + 1/m_3 - 1/m_4),$$

a result which is symmetric in m_1, m_2, m_3. This proves that the isotomic line of side 4 with respect to triangle 123 is parallel to the axis of the parabola. But the Newton line is also parallel to the axis of the parabola, for it is the locus of centers of all conics inscribed in the quadrilateral, and this locus contains the center of the parabola.

Solution to Problem 1197:
American Mathematical Monthly, **63**, (1956), 493.

A Rich Configuration

E 1197 [1956, 39]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let ABC be a right triangle and CH the altitude on the hypotenuse AB. Show that the sum of the radii of the inscribed circles of triangles ABC, HCA, HCB is equal to CH.

Solution by Leon Bankoff, Los Angeles, Calif. I. The diameter of the circle inscribed in a right triangle is equal to the sum of the legs minus the hypotenuse. Applying this relation in triangles ABC, HCA, HCB, we get

$$\frac{(AC + CB - AB) + (AH + CH - AC) + (CH + HB - CB)}{2} = CH.$$

II. In similar right triangles, the ratios of inradius to hypotenuse are equal. We may therefore write

$$r/c = r_1/b = r_2/a = (r + r_1 + r_2)/(a + b + c),$$

where r, r_1, r_2 are the inradii of triangles ABC, HCA, HCB. Since

$$r/c = CH/(a + b + c)$$

it follows that $r+r_1+r_2=CH$.

Additional selected properties of the configuration. R, S, T are the incenters of triangles AHC, CHB, ABC, respectively, and H, R', S', T' the orthogonal projections of C, R, S, T on AB. Let r, r_1, r_2 denote the inradii of triangles ABC,

AHC, CHB. P and Q are the feet of the cevians CR and CS. RS cuts AC in U and CB in V. K is the intersection of PS and RQ.

(1) T' is the circumcenter of triangle RST; Q, S, T, R, P are concyclic; $PT'=TT'=T'Q$.

(2) T is the orthocenter of triangle CRS and the circumcenter of triangle CPQ.

(3) $r_1^2+r_2^2=r^2$.

(4) $RS=CT=PT=QT=r\sqrt{2}$.

(5) Triangles HSR, ABC, AHC, HCB are similar.

(6) A, R, S, B are concyclic. Triangles RST and ABT are inversely similar.

(7) R, T', H, S are concyclic. (RS is a diameter of the circle.)

(8) $T'S$ is parallel to AC; RT' is parallel to CB.

(9) Triangles $RR'T'$ and $T'S'S$ are congruent (and similar to triangle ABC).

(10) Triangles ATB, BCS, ARC are similar.

(11) Angle $ATB=$angle $BSC=$angle $ARC=135°$.

(12) $AC=AQ$; $PB=CB$.

(13) $CU=CV$; CT is the perpendicular bisector of UV.

(14) PS, RQ, CH are concurrent at K, the orthocenter of triangle CPQ.

(15) PS is parallel to AT; RQ is parallel to TB; triangles PBS and CSB are congruent; triangles AQR and ARC are congruent.

(16) The midpoint of RS is the nine point center of triangle CPQ.

(17) The circumcircle of triangle HSR is the nine point circle of triangle CPQ.

(18) The circumcircles of triangles ARC and CSB are tangent at C, and CT is their common internal tangent.

(19) $RT=KS=SQ$; $RP=RK=TS$; triangles PKR and KQS are isosceles right triangles. (Also triangle $RT'S$.)

(20) A, P, T, C are concyclic; B, Q, T, C are concyclic.

(21) The perimeter of triangle $T'S'S=$perimeter of triangle $RR'T=CH$ (since $SS'=r_2, T'S'=r_1$, $T'S=r$).

(22) Area of triangle $RST=(a+b-c)^3/8c=r^3/c$.

(23) Area of pentagon $PQSTR=2r^3/c+r^2$.

(24) Area of triangle $CPQ=abr/c$.

Also solved by W. A. Al-Salam, L. C. Barrett, Robert Bart, G. E. Bills, R. L. Caskey, G. B. Charlesworth, N. A. Childress, T. Y. Chow, Mary Constable, R. J. Cormier, K. W. Crain, A. E. Danese, D. E. D'Atri, G. W. Day, Hazel Evans, Herta Freitag, Michael Goldberg, A. J. Goldman, Peter Gould, Cornelius Groenewoud, D. J. Hansen, Vern Hoggatt, R. T. Hood, Roger Hou, J. P. Hoyt, Raymond Huck, Louise Hutchinson, A. R. Hyde, P. W. M. John, Edgar Karst, M. S. Klamkin, W. G. Koellner, Sam Kravitz, M. A. Laframboise, L. E. Laird, B. R. Leeds, L. I. Lokomowitz, Robert Lynch, D. C. B. Marsh, Beckham Martin, C. N. Mills, C. S. Ogilvy, Margaret Olmsted, M. J. Pascual, Walter Penney, L. L. Pennisi and N. C. Scholomiti (jointly), C. F. Pinzka, P. W. A. Raine, M. A. Rachid, L. A. Ringenberg, Azriel Rosenfeld, Donald Rubin, C. M.

Sandwick, Sr., E. D. Schell, G. J. Simmons, Bernard Smilowitz, Sister M. Stephanie, A. V. Sylwester, W. R. Talbot, Chih-yi Wang, R. M. Warten, Dale Woods, Roscoe Woods, André Yandl, David Zeitlin, and the proposer. Late solutions by Paul Herzberg and Alan Wayne.

It was pointed out that this problem appears in N. A. Court, *College Geometry*, 2nd ed., p. 93, ex. 19b, and in *Scripta Mathematica*, vol. 16 (1950), p. 167.

Solution to Problem 1209:
American Mathematical Monthly, **63**, (1956), 186.

A Cross Ratio Associated with Any Triangle

E 1209 [1956, 186]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let ABC be any triangle and (I) its incircle. Let (I) touch BC, CA, AB at D, E, F, and intersect the cevians BE, CF at E', F' respectively. Show that the anharmonic ratio $D(E, F, E', F')$ is the same for all triangles ABC.

I. *Solution by W. B. Carver, Cornell University*. This is obviously a metrically special case of a more general projective theorem. The incircle may be replaced by *any conic* tangent to the sides at D, E, F, with the conic cutting the lines BE and CF at E' and F' respectively. By one of the limiting cases of Brianchon's theorem the lines AD, BE, CF meet in a point G. We set up a homogeneous coordinate system with A, B, C, G as the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$ respectively. It then follows readily that D, E, F are the points $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$; the conic has the equation

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0;$$

E', F' are the points $(1, 4, 1)$, $(1, 1, 4)$; the lines through D have the equations $kx+y-z=0$ with $k=1, -1, -3, 3$ for DE, DF, DE', DF' respectively; and the required anharmonic ratio is therefore

$$(1 + 3)(-1 - 3)/(1 - 3)(-1 + 3) = 4.$$

II. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn*. By a central projection, triangle ABC and its incircle (I) can be transformed into an equilateral triangle and its incircle. The anharmonic ratio $D(E, F, E', F')$ is invariant under this transformation and consequently is constant for all triangles. It is easy to show that $D(E, F, E', F')=4$.

Also solved by N. A. Court, P. W. M. John, D. C. B. Marsh, O. J. Ramler, Roscoe Woods, and the proposer.

Solution to Problem 1217:
American Mathematical Monthly, **64**, (1957), 45.

A Property of Euler's Function

E 1217 [1956, 342]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Evaluate

$$\prod_{d|n} d^{\phi(n/d)+\phi(d)}.$$

Solution by J. B. Johnston, Cornell University. Let f be any function defined on the integers. Then

$$\begin{aligned}\prod_{d|n} d^{f(d)+f(n/d)} &= \prod_{d|n} d^{f(d)} \prod_{d|n} d^{f(n/d)} \\ &= \prod_{d|n} d^{f(d)} \prod_{d|n} (n/d)^{f(d)} \\ &= \prod_{d|n} n^{f(d)} = n^{\sum_{d|n} f(d)}.\end{aligned}$$

Since

$$\sum_{d|n} \phi(d) = n, \tag{1}$$

the answer to the given problem is n^n.

Also solved by W. J. Buckingham, Leonard Carlitz, A. E. Danese, M. P. Drazin, L. T. Gardner, A. J. Goldman, D. S. Greenstein, Cornelius Groenewoud, Emil Grosswald, Virginia Hanly, A. R. Hyde, Richard Kelisky, Sidney Kravitz, R. G. McDermot, D. C. B. Marsh, Leo Moser, J. B. Muskat, F. R. Olson, Hiram Paly, M. Perisastri, Azriel Rosenfeld, A. V. Sylwester, Chih-yi Wang, David Zeitlin, and the proposer. Late solution by M. S. Klamkin.

Editorial Note. For a proof of (1) see, *e.g.*, Uspensky and Heaslet, *Elementary Number Theory*, p. 113. As another application of the general result established above we have

$$\prod_{d|n} d^{(n/d)+d} = n^{\sigma(n)},$$

where $\sigma(n)$ is the sum of the divisors of n.

Solution to Problem 1778:
American Mathematical Monthly, **73**, (1966), 667.

The Radii of a Triangle

E 1778 [1965, 420]. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

If R, r, r_1, r_2, r_3 are the circumradius, inradius and exradii of a triangle, prove that

$$\frac{1}{r^3}-\frac{1}{r_1^3}-\frac{1}{r_2^3}-\frac{1}{r_3^3}=\frac{12R}{r\cdot r_1\cdot r_2\cdot r_3}.$$

Solution by Ralph Schreiber, Warsaw High School, Warsaw, Ind. Denote by $\triangle$ the area of triangle ABC, by s the semiperimeter, by r_a the exradius corresponding to side a, and so forth. We recall familiar identities:

$$\triangle = rs = r_a(s-a) = r_b(s-b) = r_c(s-c) = \sqrt{rr_ar_br_c} = abc/4R.$$

Thus

$$\frac{1}{r^3}-\frac{1}{r_a^3}-\frac{1}{r_b^3}-\frac{1}{r_c^3}=\frac{1}{\triangle^3}\left[s^3-(s-a)^3-(s-b)^3-(s-c)^3\right]$$
$$= 3abc/\triangle^3 = 12R/\triangle^2 = 12R/rr_ar_br_c.$$

Also solved by A. N. Aheart, Leon Bankoff, W. J. Blundon, D. I. A. Cohen, Ragnar Dybvik (Norway), Mrs. A. C. Garstang, Michael Goldberg, Louise S. Grinstein, D. M. Hancasky, E. S. Langford, Ruth S. Lefkowitz, F. Leuenberger (Switzerland), Andrzej Makowski (Poland), D. C. B. Marsh, F. R. Prieto, J. M. Quoniam (France), S. Bhaskara Rao (India), Simeon Reich (Israel), P. A. Scheinok, Klaus Schmitt, R. Sivaramakrishnan (India), Sidney Spital, Sister M. Stephanie, M. V. Tamhankar & M. B. Suryanarayana (India), Simon Vatriquant (Belgium), C. S. Venkataraman (India), William Wernick, and the proposer.

Solution to Problem 1779:
American Mathematical Monthly, **73**, (1966), 668.

The Altitudes and Exradii of a Triangle

E 1779 [1965, 420]. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

If h_i and r_i are the altitudes and exradii of a triangle prove that

$$\frac{r_1}{h_1}+\frac{r_2}{h_2}+\frac{r_3}{h_3} \geqq 3.$$

I. *Solution by D. C. B. Marsh, Colorado School of Mines.* Since $2/h_i=1/r_j+1/r_k$ for i, j, k a cyclic permutation of 1, 2, 3 (R. A. Johnson, *Modern Geometry*, p. 189; or use the identities in E 1778 above together with $1/r=1/r_1+1/r_2+1/r_3$) it follows immediately that $\sum(r_i/h_i)=\frac{1}{2}\sum_{i\neq j}(r_i/r_j)\geqq\frac{1}{2}(6)=3$, since $(x/y)+(y/x)\geqq 2$ for positive x, y. Moreover, equality obtains only if $r_1=r_2=r_3$, i.e., the triangle is equilateral.

II. *Solution by H. Guggenheimer, University of Minnesota.* We may generalize by securing the inequality $\sum(r_i/h_i)^n\geqq 3$, $n\geqq 1$. Actually more is true: Let t_i be the lengths of the angle bisectors of the triangle. Since $t_i\geqq h_i$, the proposed inequality is weaker than

$$\sum_{i=1}^{3}\left(\frac{r_i}{t_i}\right)^m \geqq 3 \qquad m>0$$

which we now prove.

Let s be the semiperimeter of the triangle, a_i the sides. Leuenberger has proved (Elemente Math., 17 (1962) 45–46; see also 16 (1961) p. 129) that $t_i\leqq[s(s-a_i)]^{1/2}$. Hence

$$\begin{aligned}\sum\left(\frac{r_i}{t_i}\right)^m &\geqq \sum_{i\neq j\neq k}\left[\frac{s(s-a_j)(s-a_k)}{s(s-a_i)^2}\right]^{m/2}\\ &=\frac{1}{[(s-a_1)(s-a_2)(s-a_3)]^m}\sum_{j\neq k}[(s-a_j)(s-a_k)]^{3m/2}.\end{aligned}$$

The desired result now follows from the geometric-arithmetic mean inequality, and again equality holds only for the equilateral triangle.

Also solved by A. N. Aheart, Leon Bankoff, W. J. Blundon, D. I. A. Cohen, Mrs. A. C. Garstang, Michael Goldberg, H. Guggenheimer, D. M. Hancasky, E. S. Langford, F. Leuenberger (Switzerland), Andrzej Makowski (Poland), F. R. Prieto, J. M. Quoniam (France), S. Bhaskara Rao (India), Simeon Reich (Israel), P. A. Scheinok, Ralph Schreiber, R. Sivaramakrishnan (India), Sidney Spital, Sister M. Stephanie, M. V. Tamhankar & M. B. Suryanarayana (India), P. D. Thomas, Simon Vatriquant (Belgium), C. S. Venkataraman (India), and the proposer.

Makowski's student, Tadeusz Figiel observed that the required inequality is equivalent to the fact that the area of an orthic triangle is not greater than one-quarter of the area of a given (acute-angled) triangle. [*Proof:* Let ABC be an orthic triangle of $A_1B_1C_1$. Then A_1, B_1, C_1 are the centers of ex-circles and the ratio of areas of ABC_1 and ABC is equal to the ratio of altitudes on the common side AB, i.e., r_3/h_3.]

Solution to Problem 1877:
American Mathematical Monthly, **74**, (1967), 869.

Convex Pentagon Inscribed in a Semicircle

E 1877 [1966, 410]. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey*

Let $ABCDE$ be a convex pentagon inscribed in a unit circle with AE as diameter, and let $AB=a$, $BC=b$, $CD=c$, $DE=d$. Then prove that

$$a^2+b^2+c^2+d^2+abc+bcd<4.$$

Solution by Allan Wachs, Student at Far Rockaway (N. Y.) High School. Draw AC and CE and put $\theta=\sphericalangle CEA$. Then $\sphericalangle CAE=90°-\theta$, $\sphericalangle CBA=180°-\theta$, $\sphericalangle CDE=180°-(90°-\theta)=90°+\theta$. By the law of cosines,

$$AC^2=a^2+b^2-2ab\cos(180°-\theta)=a^2+b^2+2ab\cos\theta, \tag{1}$$

$$CE^2=c^2+d^2-2cd\cos(90°+\theta)=c^2+d^2+2cd\sin\theta. \tag{2}$$

From the right triangle ACE, $AC^2+CE^2=AE^2=4$, also $AC=2\sin\theta>b$ and $CE=2\cos\theta>c$ (because of the obtuse angles). Substitution of these results into the sum of (1) and (2) gives at once

$$4>a^2+b^2+c^2+d^2+abc+bcd.$$

Also solved by Leon Bankoff, W. J. Blundon, L. Carlitz, Mannis Charosh, M. A. Ettrick, Michael Goldberg, M. G. Greening (Australia), Ned Harrell, Donald Jeffords, Erwin Just, J. D. E. Konhauser, Dan Marcus, Lieselotte Miller, Norman Miller, C. B. A. Peck, Al Somoyajulu, J. L. Standig, C. S. Venkataraman (India), J. C. Williams, Dale Woods, and the proposer.

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From the right triangle ACE, $AC^2+CE^2=AE^2=4$, also $AC=2\sin\theta>b$ and $CE=2\cos\theta>c$ (because of the obtuse angles). Substitution of these results into the sum of (1) and (2) gives at once

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Also solved by Leon Bankoff, W. J. Blundon, L. Carlitz, Mannis Charosh, M. A. Ettrick, Michael Goldberg, M. G. Greening (Australia), Ned Harrell, Donald Jeffords, Erwin Just, J. D. E. Konhauser, Dan Marcus, Lieselotte Miller, Norman Miller, C. B. A. Peck, Al Somoyajulu, J. L. Standig, C. S. Venkataraman (India), J. C. Williams, Dale Woods, and the proposer.

Solution to Problem 2100:
American Mathematical Monthly, **76**, (1969), 563.

Six Relations

E 2100 [1968, 670]. *Proposed by H. Demir, Middle East Technological University, Ankara, Turkey*

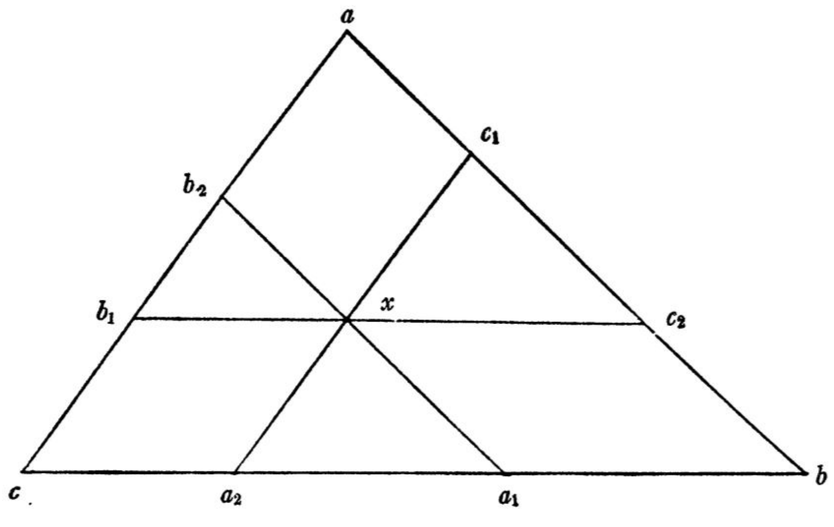
Show that any five of the relations

$$(1)\ \frac{x-a_1}{a_1-a_2}=\frac{a-b}{b-c},\qquad (2)\ \frac{x-b_1}{b_1-b_2}=\frac{b-c}{c-a},\qquad (3)\ \frac{x-c_1}{c_1-c_2}=\frac{c-a}{a-b},$$

$$(4)\ x+a=b_2+c_1,\qquad (5)\ x+b=c_2+a_1,\qquad (6)\ x+c=a_2+b_1$$

imply the sixth. Interpret this set of consistent relations geometrically letting a, b, c be the affixes, in the complex plane, of a triangle of reference ABC and other numbers be those of other points.

Solution by Michael Goldberg, Washington, D. C. Take any triangle, represented by the vertices a, b, c. Take any point x in the plane of the triangle. Draw parallels to the sides of the triangle through the point x. Let the intersections of these parallels with the sides be a_1, a_2, b_1, b_2, c_1, c_2, as shown in the figure. Then the six given relations hold. If one of the six points, say c_2, is omitted, then x



can be found as the intersection of a_2c_1 with b_2a_1, and then c_2 is determined as the intersection of b_1x with ab.

Also solved by A. F. Gentzel, Jr., Simeon Reich (Israel), and the proposer.

Solution to Problem 2101:
American Mathematical Monthly, **76**, (1969), 564.

Three Parabolas and a Triangle

E 2101 [1968, 670]. *Proposed by H. Demir, Middle East Technological University, Ankara, Turkey*

ABC is a triangle. Let P_a denote the parabola tangent to the sides AB, AC at B, C respectively. The parabolas P_b and P_c are similarly defined. Let these parabolas intersect at the points A', B', C' inside ABC. Denote the areas of the (curvilinear) triangular regions ABC, $A'B'C'$, $AB'C'$, $BC'A'$, $CA'B'$, $A'BC$, $B'CA$, $C'AB$ by Δ, Δ_0, Δ_a', Δ_b', Δ_c', Δ_a'', Δ_b'', Δ_c''. Then prove

$$(1) \qquad \Delta_a' = \Delta_b' = \Delta_c'(\equiv \Delta_1), \qquad \Delta_a'' = \Delta_b'' = \Delta_c''(\equiv \Delta_2),$$

$$(2) \qquad \Delta_0 : \Delta_1 : \Delta_2 : \Delta = 15:17:5:81.$$

Solution by the proposer. Under parallel projections the nature of conics, the tangency and ratios of segments and areas are invariant, and any triangle can be transformed into an equilateral triangle. Hence it will suffice to prove the assertion for an equilateral triangle. So, part (1) is already proved.

To prove (2), let ABC be an equilateral triangle located in the coordinate plane such that $A = (1, \sqrt{3})$, $B = (0, 0)$, $C = (2, 0)$. The equations of parabolas P_a and P_c are found to be

$$(1) \qquad P_a:\ y = (x - \tfrac{1}{2}x^2)\sqrt{3},$$

$$(2) \qquad P_c:\ \sqrt{3}\,x^2 + 6xy + 3\sqrt{3}\,y^2 - 16y = 0.$$

From (1) and (2) we obtain

$$(3) \qquad \Delta_0 + 2\Delta_1 + \Delta_2 = \sqrt{3}\int_0^2 (x - \tfrac{1}{2}x^2)dx = \tfrac{2}{3}\sqrt{3} = \tfrac{2}{3}\Delta,$$

$$(4) \qquad y = \frac{8\sqrt{3}}{9} - \frac{\sqrt{3}}{3}x - \frac{4\sqrt{3}}{9}\sqrt{4 - 3x} \qquad (0 \leqq x \leqq \tfrac{4}{3}).$$

We find, therefore,

$$(5) \qquad \Delta_2 = 2\int_0^1 y\,dx = \frac{5\sqrt{3}}{81} = \frac{5}{81}\Delta,$$

with

$$(6) \qquad \Delta_0 + 3\Delta_1 + 3\Delta_2 = \Delta.$$

Solving the system (3), (5), (6) for Δ_0, Δ_1, Δ_2, we get the required result.

Also solved by Anders Bager (Denmark), Jordi Dou (Spain), Michael Goldberg, M. G. Greening (Australia), Norman Miller, J. M. Quoniam (France), and A. Zujus.

Solution to Problem 2109:
American Mathematical Monthly, **76**, (1969), 698.

Triangle Construction

E 2109 [1968, 780]. *Proposed by H. Demir, Middle East Technical University, Ankara, Turkey*

Let ABC be a triangle and A' be any fixed point on the side BC. Construct the inscribed triangle $A'B'C'$ which is directly similar to a given triangle XYZ.

Note by A. W. Walker, Toronto, Canada. The required construction will be found in N. A. Court, *College Geometry*, ed. 1, 1925, p. 47. It is a simple application of the following theorem, established on p. 46: *If one vertex of a triangle of variable size and given shape remains fixed and a second vertex moves on a given straight line, then the locus of the third vertex is also a straight line.*

Also solved by Anders Bager (Denmark), Walter Bluger, C. W. Eliason, Jr., Michael Goldberg, M. G. Greening (Australia), Beckham Martin, D. N. Page, and the proposer.

Solution to Problem 2110:
American Mathematical Monthly, **76**, (1969), 698.

Similar Triangles

E 2110 [1968, 780]. *Proposed by H. Demir, Middle East Technical University, Ankara, Turkey*

If, in a plane, the triangles AUV, VBU, UVC are directly similar to a given triangle, then so is ABC.

Solution by M. G. Greening, University of New South Wales, Australia. Represent the points by complex numbers using the appropriate lower case letters and take the given triangle as $Z_1Z_2Z_3$. Let the direct similarities be $z \to \alpha_i z + \beta_i$ $(i=1, 2, 3)$. Then $u = \alpha_i z_{i+1} + \beta_i$, $v = \alpha_i z_{i+2} + \beta_i$ $(i=1, 2, 3)$, taking subscripts modulo 3. Then $\alpha_i z_i (z_{i+1} - z_{i+2}) = z_i(u - v)$ and

$$\beta_i(z_{i+1} - z_{i+2}) = z_{i+1}v - z_{i+2}u$$

so that

$$\sum_{i=1}^{3} (\alpha_i z_i + \beta_i)(z_{i+1} - z_{i+2}) = 0.$$

As $\sum_{i=1}^{3} (z_{i+1} - z_{i+2}) = 0$ and $\sum_{i=1}^{3} z_i(z_{i+1} - z_{i+2}) = 0$ we get

$$0 = \begin{vmatrix} \alpha_1 z_1 + \beta_1 & z_1 & 1 \\ \alpha_2 z_2 + \beta_2 & z_2 & 1 \\ \alpha_3 z_3 + \beta_3 & z_3 & 1 \end{vmatrix} = \begin{vmatrix} a & z_1 & 1 \\ b & z_2 & 1 \\ c & z_3 & 1 \end{vmatrix}$$

which is a sufficient condition for a direct similarity: $z_1 \to a$, $z_2 \to b$, $z_3 \to c$ to exist.

Also solved by Leon Bankoff, Jordi Dou (Spain), C. W. Eliason, Jr., Michael Goldberg, Norman Miller, Simeon Reich (Israel), A. W. Walker, and the proposer.

Walker points out that the result may be found on p. 289 of R. A. Johnson, *Modern Geometry* (1929).

Solution to Problem 2124:
American Mathematical Monthly, **76**, (1969), 938.

All Triangles Generate Right Triangles

E 2124 [1968, 899]. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey*

Construct on the sides BC, CA, AB of a triangle ABC, exteriorly, the squares $BCDE$, $ACFG$, $BAHK$ and build parallelograms $FCDQ$, $EBKP$. Show that APQ is an isosceles right triangle.

Solution by W. E. Buker, Pittsburgh Public Schools. Assign coordinates $A(0, 0)$; $B(a, 0)$; $C(b, c)$. Then find by inspection the coordinates $F(b-c,\ b+c)$, $D(b+c,\ c+a-b)$, $Q(b,\ a+c)$, $K(a,\ -a)$, $E(a+c,\ a-b)$, $P(a+c,\ -b)$. Since AQ and AP have equal lengths and are perpendicular, the theorem follows.

Also solved by forty other readers.

Note. It follows at once that if parallelogram $HAGR$ is constructed, then BQR and CRP are also isosceles right triangles.

A. W. Walker points out that an extensive investigation of triangles bordered by squares is found in a paper by Musselman, this MONTHLY, 43 (1936), 539–548.

Solution to Problem 2160:
American Mathematical Monthly, **76**, (1969), 1146.

Some New Triangle Inequalities

E 2160 [1969, 300]. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

Let p_i, x_i be the distances of an interior or a boundary point P of a triangle $A_1A_2A_3$ from the vertex A_i and from the side opposite to A_i, $i=1, 2, 3$, with r the inradius. Prove the inequalities

(a) $$\sum_{i=1}^{3} p_i(\tfrac{1}{2}\sin A_i) \leqq \sum_{i=1}^{3} x_i \leqq \sum_{i=1}^{3} p_i \sin(\tfrac{1}{2} A_i).$$

(b) $$p_2p_3 + p_3p_1 + p_1p_2 \geqq 8x_1x_2x_3/r.$$

Solution by M. G. Greening, University of New South Wales, Australia. Let a_i be the side opposite A_i, let P_i be the angle $A_{i-1}PA_{i+1}$, $B_{i,j}$ be the angle between p_i and a_j at A_i, so that $B_{i,i+1}+B_{i,i-1}=A_i$. (All additions of subscripts are modulo 3.) Then $x_i=p_{i+1}\sin B_{i+1,i}=p_{i-1}\sin B_{i-1,i}$ and

$$\sum_i x_i = \tfrac{1}{2}\sum_j p_j(\sin B_{j,j+1} + \sin B_{j,j-1})$$

$$= \tfrac{1}{2}\sum_j p_j\cdot 2\sin(\tfrac{1}{2}A_j)\cos\tfrac{1}{2}(B_{j,j+1} - B_{j,j-1}).$$

The inequality $0 \leqq |B_{j,j+1}-B_{j,j-1}| \leqq A_j$ then yields (a).

As $p_1p_2 \sin P_3 = x_3a_3$, we obtain

$$\sum_i p_ip_{i+1} = \sum_i \frac{x_ia_i}{\sin P_i} \geqq \frac{3\left(\prod_i a_i\cdot\prod_i x_i\right)^{1/3}}{\left(\prod_i \sin P_i\right)^{1/3}}$$

$$\geqq 3\left(\prod_i a_i \prod_i x_i\right)^{1/3}\left(\prod_i \sin P_i\right)^{-1/3} \geqq 2\sqrt{3}\left(\prod_i a_i \cdot \prod_i x_i\right)^{1/3}.$$

The last statement follows from the fact that $\prod_i \sin P_i$ with $\sum_i P_i = 2\pi$ has a maximum when $P_1 = P_2 = P_3$.

For (b) we now show

$$\text{(i)} \qquad 2\sqrt{3}r\left(\prod_i a_i\right)^{1/3} \geqq 8\left(\prod_i x_i^2\right)^{1/3}.$$

As $\sum_i x_i = 2\Delta$, $\prod_i x_i$ has a maximum when $a_1x_1 = a_2x_2 = a_3x_3 = 2\Delta/3$, so that max $8(\prod_i x_i^2)^{1/3} = 2^5\Delta^2 3^{-2}(\prod_i a_i)^{-2/3}$. (i) will follow if $3^{5/2}r \cdot \prod_i a^i \geqq 2^4\Delta^2$, or

$$\text{(ii)} \qquad 3^{5/2}R \geqq 4s,$$

as $\prod_i a_i = 4R\Delta$, where R is the circumradius. But the triangle of largest perimeter which can be inscribed in a given circle is equilateral and the inequality (ii) is certainly true then, so that (b) is established. In fact, 8 could be replaced by 12 in (b).

Also solved by Simeon Reich (Israel), T. Tamura (Japan), C. S. Venkataraman (India), A. W. Walker and the proposer.

The improved inequality for part (b) was conjectured by Walker and proved by Reich. It is interesting to note that aside from a solution to part (a) by L. Carlitz, all solvers and the proposer reside outside the United States of America.

Solution to Problem 2213:
American Mathematical Monthly, **77**, (1970), 1109.

Quadrilaterals with the Nagel Property

E 2213 [1970, 79]. *Proposed by H. Demir, Middle East Technical University, Turkey*

Let us say that a (planar) polygon has the *Nagel property* if the lines through the vertices of the polygon and bisecting the perimeter of the polygon are concurrent. It is known that all triangles have the Nagel property and that not all quadrilaterals have the property. Determine the simple nondegenerate quadrilaterals that have the Nagel property.

Solution by the editor based on the proposer's solution. Let $ABCD$ be a quadrilateral having the Nagel property. Let each of the lines AA', BB', CC', DD' bisect the perimeter and pass through the Nagel point N. Let $AB=a$, $BC=b$, $CD=c$, $DA=d$. We may suppose $a+b\geqq c+d$ and $b+c\geqq a+d$. It then follows that C' lies on segment AB, B' on CD, and A' and D' on BC (Fig. 1). Set $BD'=u$,

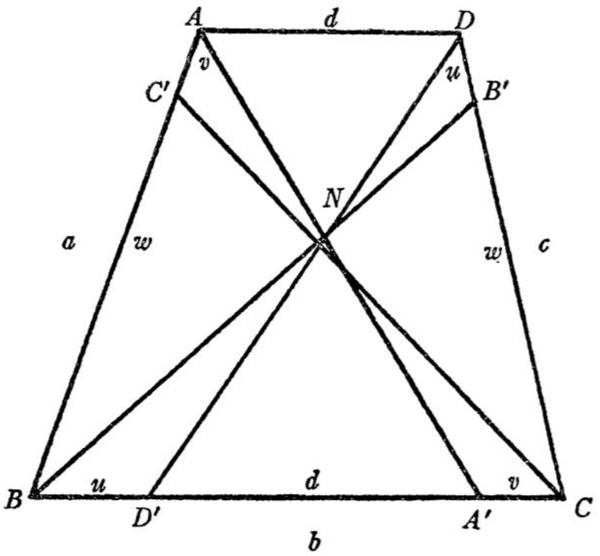


Fig 1

$A'C=v$, $CB'=w$, and $a+b+c+d=2s$. Since $BC+CB'=s=D'C+CD$, then $DB'=u$. Similarly $AC'=v$, $BC'=w$ and $D'A'=d$. We distinguish three cases.

CASE 1. $u\neq 0$ and $v\neq 0$. Thinking of AA' and DD' meeting at N, we see that a necessary and sufficient condition for CC' to pass through N is

$$\frac{v}{w}\cdot\frac{b}{-v}\cdot\frac{A'N}{NA} = -1$$

by Menelaus' theorem applied to triangle ABA' cut by line CNC'. Similarly, BB' passes through N if and only if

$$\frac{-u}{b}\cdot\frac{w}{u}\cdot\frac{DN}{ND'} = -1,$$

using triangle $DD'C$ cut by BNB'. Then

$$\frac{DN}{ND'} = \frac{b}{w} = \frac{AN}{NA'}\,.$$

Hence AD is parallel to $D'A'$, and since these segments are also equal, it follows that $ADA'D'$ is a parallelogram. The diagonals AA' and DD' then bisect each other, so $AN=NA'$ and $b=w$. But $b+w=s$, so $2w=a-b+c+d=2b$, from which we obtain

$$b = \frac{a+c+d}{3}\,.$$

Hence $ABCD$ is a trapezoid such that the longer of its two parallel bases is the arithmetic mean of its other three sides. Reversing the argument of this paragraph shows that every such trapezoid has the Nagel property. For example, the trapezoid with vertices (0, 0), (19, 0), (6, 24), (0, 24) has $N=(7, 12)$.

CASE 2. $u=v=0$. Then $a+b=c+d=s$ and $a+d=b+c=s$, so $2a+b+d=2c+b+d$. Hence $a=c$ and $b=d$, so $ABCD$ is a parallelogram. Clearly every parallelogram has the Nagel property.

CASE 3. $u=0$ and $v\neq 0$ (or vice versa). Let AC and BD meet at M (Fig. 2). Now $a+d=b+c=s$, so $d+v+w=C'B+BC=CD+DA+AC'=c+d+v$, whence $c=w$. Applying Menelaus' theorem to triangle ABA' cut by CNC' and to triangle $AA'C$ cut by BNM, we obtain

$$\frac{b}{-v}\cdot\frac{A'N}{NA}\cdot\frac{v}{c} = -1, \qquad \text{so } \frac{A'N}{NA} = \frac{c}{b},$$

$$\frac{AN}{NA'}\cdot\frac{-d}{b}\cdot\frac{CM}{MA} = -1, \qquad \text{so } \frac{CM}{MA} = \frac{b}{d}\cdot\frac{A'N}{NA} = \frac{b}{d}\cdot\frac{c}{b} = \frac{c}{d}\,.$$

Since M divides side CA of triangle DAC in the ratio c/d of the adjacent sides, then DM bisects angle D. Hence $\sphericalangle ADM=\sphericalangle MDC=\alpha$. Applying the law of

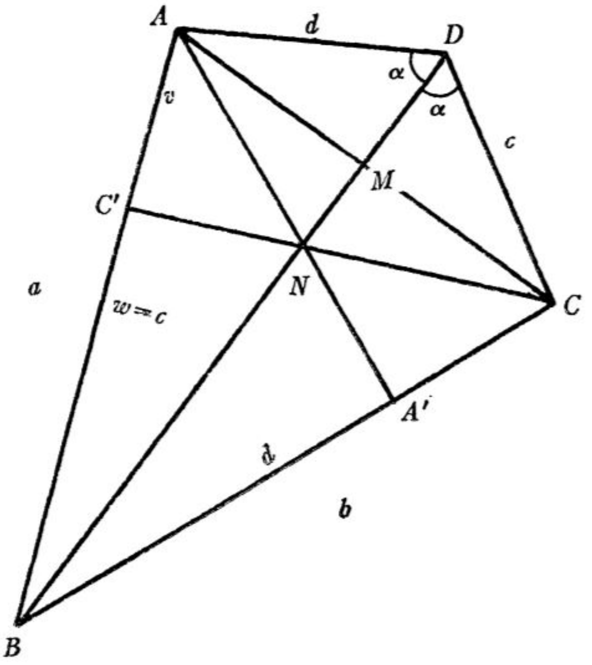


FIG. 2

cosines to triangles ABD and CBD and letting $BD=f$, we have

$$a^2 = d^2 + f^2 - 2df \cos \alpha,$$
$$b^2 = c^2 + f^2 - 2cf \cos \alpha,$$

so

$$a^2 - b^2 = d^2 - c^2 - 2f(d - c) \cos \alpha.$$

Assuming $d \geqq c$, let $d=c+e$, so $b=c+v+e$ and $a=c+v$. Now $a^2-(a+e)^2=(c+e)^2-c^2-2ef \cos \alpha$, which simplifies to

$$ef \cos \alpha = ce + ae + e^2.$$

If $e \neq 0$, then $\cos \alpha=(c+a+e)/f=(c+b)/f$, an impossibility since $c+b>f$. Hence $e=0$, so $c=d$ and $a=b$. The figure is therefore a kite. By symmetry, every kite has the Nagel property.

Three other correspondents mentioned the kite and the parallelogram.

Solution to Problem 2311:
American Mathematical Monthly, **79**, (1972), 777.

The Compleat Cyclic Quadrilateral

E 2311 [1971, 793]. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey*

Prove that, if a quadrilateral $A_1A_2A_3A_4$ can be inscribed in a circle, then the (six) lines drawn from the midpoints of A_pA_q perpendicular to A_rA_s (p, q, r, s distinct) are concurrent.

Solution by Sister Stephanie Sloyan, Georgian Court College, Lakewood, N.J. Assume that the circle is the unit circle and identify the point A_i with the complex number a_i in the usual manner. Then the line from the midpoint of the segment A_pA_q perpendicular to A_rA_s is given by

$$z - a_r a_s \bar{z} = \tfrac{1}{2}(a_p + a_q)\ \frac{a_p a_q - a_r a_s}{a_p a_q},$$

and it is easily calculated that all six lines pass through the point $\frac{1}{2}(a_1 + a_2 + a_3 + a_4)$. J. W. Clawson, *The complete quadrilateral*, Annals of Math. 20 (1918–1919), 232–261, calls this point the *orthic center* of the quadrilateral.

In a similar fashion one can show that the three lines joining the midpoint of A_pA_q to that of A_rA_s (p, q, r, s distinct) are each bisected by a point identified by Clawson as the *mean center* of the quadrilateral. Since the mean center is given by $\frac{1}{4}(a_1 + a_2 + a_3 + a_4)$, it follows that it lies halfway between the orthic center and the circumcenter.

Also solved by Michael Goldberg, Leonard Goldstone, M. G. Greening (Australia), N. G. Gunderson, V. F. Ivanoff, Lew Kowarski, Harry Lass, O. P. Lossers (Netherlands), Rick Troxel, and the proposer.

Editorial Note. This theorem and its solution appear on page 59 of Yaglom, *Complex Numbers in Geometry*, Academic Press, 1968, along with many other interesting properties of cyclic quadrilaterals, cyclic pentagons, etc. (see pages 54–68). The point of concurrence of this problem is called the *anticenter* by Lucien Droussent (*On a theorem of J. Griffiths*, this MONTHLY, 54 (1947), 538–540). The anticenter N is the midpoint of the quadrilateral's *Euler segment* which joins its circumcenter O to the center H of the circle through the four orthocenters H_m of the triangles $A_iA_jA_k$ ($\{i, j, k, m\} = \{1, 2, 3, 4\}$); these orthocenters form a quadrilateral congruent to the given one and symmetric to it in point N. Furthermore, the eight points A_i and H_i lie by fours on four distinct pairs of circles, each pair having N as center of symmetry.

The eight congruent nine-point circles for the four triangles $A_iA_jA_k$ and four triangles $H_iH_jH_k$ all pass through N, and their centers lie on another congruent circle centered at N. Thus N can be called the *eight circle* point and this last circle the *eight point circle* for the quadrilateral.

There are four distinct Simson lines for the eight points A_m with triangles $A_iA_jA_k$ and H_m with triangles $H_i H_j H_k$, and these Simson lines all pass through N. In fact, one can form 280 (180 of which are distinct) pedal circles (and lines) by taking any one of these eight points with the triangle formed by any three others, and all of them pass through N.

The nine point centers N_m for the four triangles $A_iA_jA_k$ form a quadrilateral homothetic to $H_1H_2H_3H_4$ in center O with ratio $\frac{1}{2}$, hence homothetic to $A_1A_2A_3A_4$ in center G, 1/3 of the way from O to H, with ratio $-\frac{1}{2}$. Similarly, the nine-point centers N'_m for the triangles $H_iH_jH_k$ are homothetic to $H_1H_2H_3H_4$ in center G', 2/3 of the way from O to H, with ratio $-\frac{1}{2}$. Their common circumcircle has center N and radius half the given quadrilateral's circumradius. In a similar manner (see E 1740 [1965, 1026]) the centroids G_m for the triangles $A_iA_jA_k$ form a quadrilateral homothetic to $H_1H_2H_3H_4$ in center O with ratio 1/3, hence homothetic to $A_1A_2A_3A_4$ in center S (the mean center) 1/4 of the way from O to H, with ratio $-\frac{1}{3}$. Its circumcenter is G. Similarly, the centroids G'_m (whose circumcenter is G') for the triangles $H_iH_jH_k$ determine the other quadrisection point S' of OH. Furthermore, N is the center of symmetry for the two quadrilaterals $N_1N_2N_3N_4$ and $N'_1N'_2N'_3N'_4$ and also for $G_1G_2G_3G_4$ and $G'_1G'_2G'_3G'_4$.

There are eight *orthocentroidal circles* (see Droussent) on the segments G_iH_i and on $G'_iH'_i$ as diameters, pairs of which determine 16 distinct radical axes all passing through N, so N is the center of a circle orthogonal to all these eight circles.

We see that the Euler segment could well be renamed the *seven point line* (points O, S, G, N, G', S', H). With this notation, since points G and N trisect and bisect OH, the resemblance to the Euler line of a triangle is striking.

See also H. G. Forder, *Higher Course Geometry*, Cambridge University Press, 1949, 232–235, and R. A. Johnson, *Modern Geometry*, Houghton-Mifflin, 1929, pp. 169, 207, 243, and 251–253.

Solution to Problem 2312:
American Mathematical Monthly, **79**, (1972), 778.

An Application of Ceva's Theorem

E 2312 [1971, 793]. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey*

Let D be a point in the plane of a positively oriented triangle ABC and let AD, BD, CD intersect the respective opposite sides in A_1, B_1, C_1. If the oriented segments $\overline{BA_1}$, $\overline{CB_1}$, $\overline{AC_1}$ are equal ($=\delta$), then D is uniquely determined and lies in the interior of ABC. (Notice the analogy between D and the Brocard point Ω.)

Solution by Michael Goldberg, Washington, D.C. Let the lengths of the sides of the triangle be a, b, c, where $a \leqq b \leqq c$. Then by Ceva's Theorem, we have the equation

$$(a-\delta)(b-\delta)(c-\delta) = \delta^3. \tag{*}$$

The left member of (*) is a function which decreases monotonically from abc at $\delta = 0$ to zero at $\delta = a$, and the right member is a function which increases monotonically from zero at $\delta = 0$. Hence the two functions are equal for exactly one real value of δ which lies in the interval $(0, a)$; it is easy to see also that there are no other real solutions to (*).

Note that if, instead, the segments CA_1, BC_1, and AB_1 are equal, then the value of δ is the same, but the transversals cross at another point E. The points D and E coincide only for the equilateral triangle.

Also solved by Bernhard Andersen (Denmark), Harold Donnelly, Jordi Dou (Spain), M.G. Greening (Australia), V. F. Ivanoff, and the proposer.

Editor's Comment. L. Goldstone located a complete discussion of this point, its isotomic conjugate, and their properties in Peter Yff, *An analogue of the Brocard Points*, this MONTHLY 70 (1963), 495–501.

Solution to Problem 2363:

American Mathematical Monthly, **80**, (1973), 694.

On Spherical Triangles

E 2363 [1972, 663]. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

Characterize pairs of spherical triangles ABC and $A'B'C'$ for which $A' = a$, $B' = b$, $C' = c$, $A = a'$, $B = b'$, $C = c'$.

Solution by M. G. Greening, University of New South Wales, Australia. For any spherical triangle $A'B'C'$ we have:

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A \tag{1}$$

and the two others following from the permutations (a, b, c), (a, c, b). So

$$\cos A = \cos B \cos C + \sin B \sin C \cos a, \tag{2}$$

and so on. But from consideration of the polar triangle of ABC,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a. \tag{3}$$

Then $\cos B \cos C = \cos C \cos A = \cos A \cos B = 0$ and we have, say, $A = B = \pi/2$, yielding $a = b = \pi/2$ from (2). Also $\cos C = \cos c$, which must give $C = c$ as $c > 0$, $C < \pi$. Consequently

$$A' = B' = a' = b' = \pi/2 \text{ and } c' = C' = c = C,$$

so that the two triangles are necessarily congruent.

Also solved by Michael Goldberg, Lew Kowarski, Clellie Oursler & Eric Sturley, and the proposer.

Solution to Problem 2462:

American Mathematical Monthly, **92**, (1985), 360.

The Extended Erdős-Mordell Inequality

E 2462 [1974, 281]. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let P be a point interior to the triangle $A_1A_2A_3$. Denote by R_i the distance from P to the vertex A_i, and denote by r_i the distance from P to the side a_i opposite to A_i. The Erdős-Mordell inequality asserts that

$$R_1 + R_2 + R_3 \geqslant 2(r_1 + r_2 + r_3).$$

Prove that the above inequality holds for every point P in the plane of $A_1A_2A_3$ when we make the interpretation $R_i \geqslant 0$ always and r_i is positive or negative depending on whether P and A_i are on the same side of a_i or on opposite sides.

Editorial note: Professor Clayton W. Dodge, Department of Mathematics, University of Maine refereed the "solutions" submitted to this problem in 1974 and found that there were no solutions. Since that time Professor Dodge himself has solved the problem. His solution appears in *Crux Mathematicorum*, vol. 10, no. 9, November 1984, pages 274–281.

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THE EXTENDED ERDÖS-MORDELL INEQUALITY

CLAYTON W. DODGE

Ten years ago *The American Mathematical Monthly* published the following Problem E 2462 [81 (1974) 281], which is an extension of the earlier Problem 3740 proposed by Paul Erdös [42 (1935) 396] and first solved by L.J. Mordell [44 (1937) 252-254]:

"E 2462. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let P be a point interior to the triangle $A_1A_2A_3$. Denote by R_i the distance from P to the vertex A_i, and denote by r_i the distance from P to the side a_i opposite to A_i. The Erdös-Mordell inequality asserts that

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3).$$

Prove that the above inequality holds for every point P in the plane of $A_1A_2A_3$ when we make the interpretation $R_i \geq 0$ always and r_i is positive or negative depending on whether P and A_i are on the same side of a_i or on opposite sides."

It was my pleasure in 1974 to referee the solutions to this problem. Curiously, each of the solvers started with the solution to the original Erdös inequality given by Kazarinoff [1] and modified it for the case where r_1, r_2, or r_3 is negative. Each made the same error, invalidating the proof. Curiously, Kazarinoff stated that his proof "holds even if P lies outside the triangle, provided it remains inside the circumcircle", but the Elementary Problem Department editors could not see that such an extension of the proof was possible without committing the same error the other solvers had made. We outline Kazarinoff's proof and describe the error. Since we shall rely heavily on this proof, our outline is quite complete. It is interesting to note that, if Kazarinoff's statement could have been verified then, a proof would have been published in 1975.

In Demir's notation, Kazarinoff let P lie within angle A_1 and then he reflected triangle $A_1A_2A_3$ in the bisector A_1T of angle A_1 into triangle $A_1A_2'A_3'$, as shown in Figure 1. Noting that the bisector of angle A_1 also bisects the angle between the altitude A_1D and the circumradius OA_1, he used a theorem of Pappus which states that the area of the parallelogram whose adjacent sides are A_1A_2' and A_1P plus the area of the parallelogram whose adjacent sides are A_1P and A_1A_3' is equal to the area of the parallelogram erected on $A_2'A_3'$ whose sides emanating from A_2' and A_3' are equal, as vectors, to $\vec{A_1P}$. Since $A_1P = R_1$, Kazarinoff obtained the first of equalities (1), and the other two are obtained in the same way when P lies within angles A_2 and A_3:

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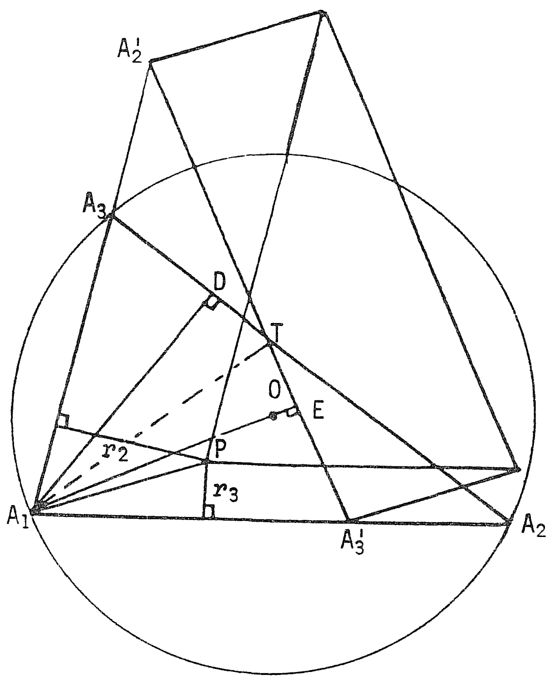

Figure 1

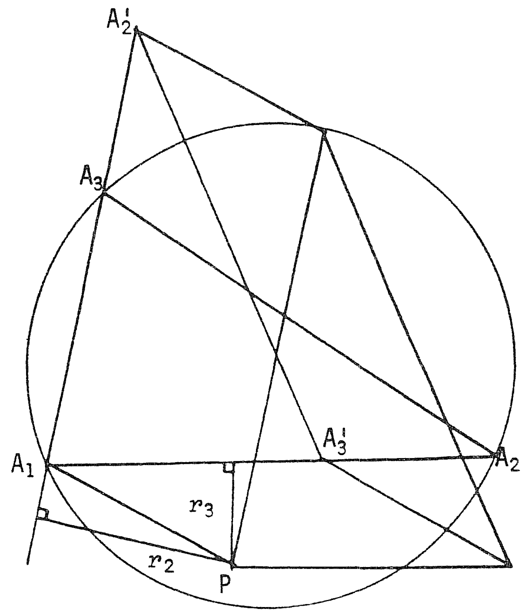

Figure 2

$$\begin{cases} a_1R_1 \cos(OA_1P) = a_2r_3 + a_3r_2 \\ a_2R_2 \cos(OA_2P) = a_3r_1 + a_1r_3 \\ a_3R_3 \cos(OA_3P) = a_1r_2 + a_2r_1. \end{cases} \tag{1}$$

From this follows, when P is an interior point of the triangle,

$$\begin{cases} R_1 + R_2 + R_3 \geq R_1 \cos(OA_1P) + R_2 \cos(OA_2P) + R_3 \cos(OA_3P) \\ \qquad = (\frac{a_2}{a_3}+\frac{a_3}{a_2})r_1 + (\frac{a_3}{a_1}+\frac{a_1}{a_3})r_2 + (\frac{a_1}{a_2}+\frac{a_2}{a_1})r_3 \\ \qquad \geq 2(r_1 + r_2 + r_3) \end{cases} \tag{2}$$

because $x + 1/x \geq 2$ when $x > 0$.

Equations (1) hold for all locations of P, provided Demir's sign convention is observed. Then also the first two lines of (2) hold. We shall make use of this in our proofs later in this paper, so a proof is presented.

Let P lie outside angle A_1 and outside angle A_2 but inside angle A_3 and inside the circumcircle of triangle $A_1A_2A_3$, as shown in Figure 2. Using the notation of Figure 1, we see that the parallelogram on side $A_2'A_3'$ now is the difference between those on sides A_1A_2' and A_1A_3'. Accordingly,

$$a_1R_1 \cos(OA_1P) = -a_2r_3 + a_3r_2,$$

where we take the r_i all nonnegative; similarly,

$$a_2R_2 \cos(OA_2P) = a_3r_1 - a_1r_3,$$

and we have as before

$$a_3R_3 \cos(OA_3P) = a_1r_2 + a_2r_1,$$

since P lies within angle A_3. Thus equations (1) are true for this case if we observe Demir's sign convention. That they also hold in other cases is not needed here. Since the cosines of the angles OA_iP are all still positive because P lies inside the circumcircle, the first two lines of (2) both still hold. Only the third line of (2) is in doubt. In fact, Kazarinoff's argument fails at this point, as explained in the next paragraph.

The error in the submitted solutions to Problem E 2462 occurred when one of the r_i, say r_3, is negative. Then we still have

$$\frac{a_1}{a_2} + \frac{a_2}{a_1} \geq 2,$$

but, since $r_3 < 0$, the inequality reverses to give

$$\left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right)r_3 \leq 2r_3,$$

rendering the argument inconclusive. The editors could find no simple remedy for this flaw since the extended theorem requires that either one or two of the distances r_i be negative. We wrote to those who had submitted solutions, and Leon Bankoff and I corresponded for perhaps a year in attempting to put together a satisfactory proof. Over the next nine years I returned to the problem from time to time, fascinated by its challenge.

Two cases were disposed of almost immediately.

Case 1. Point P lies inside the angle vertical to a vertex angle.

For example, let P lie inside the angle vertical to A_1, as shown in Figure 3. Then r_2 and r_3 are to be taken negative and we must prove that

$$R_1 + R_2 + R_3 \geq 2(r_1 - r_2 - r_3),$$

where the r_i have all been taken nonnegative and we have inserted the appropriate negative signs. Because R_2 and r_1 are hypotenuse and leg of a right triangle, we have

$$R_2 \geq r_1, \qquad \text{and similarly} \qquad R_3 \geq r_1.$$

Thus

$$R_1 + R_2 + R_3 \geq R_2 + R_3 \geq r_1 + r_1 \geq 2(r_1 - r_2 - r_3). \quad \square$$

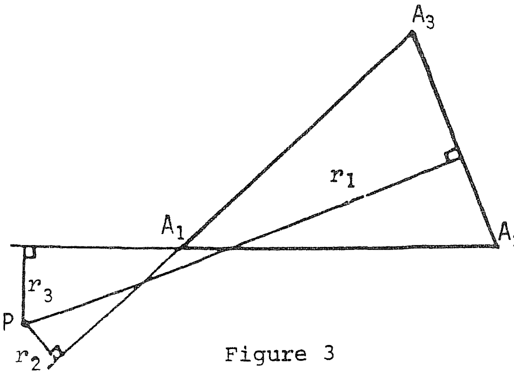

Figure 3

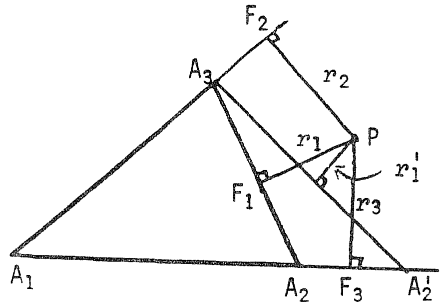

Figure 4

Case 2. Point P is interior to an angle of the triangle, but far enough outside the triangle so that a foot F_i of a distance r_i lies outside the triangle.

As shown in Figure 4, we take P lying within angle A_1 and far enough outside the triangle so that, say, the foot F_3 of distance r_3 lies outside the triangle. Then r_1 is taken negative. Choose point A_2' so that F_3 is the midpoint of segment A_2A_2'. Then $PA_2 = PA_2'$ and the three distances R_i for triangle $A_1A_2A_3$ are the same as those for triangle $A_1A_2'A_3$. Also r_2 and r_3 remain unchanged, and only r_1 changes to r_1'. If, as pictured, P lies outside triangle $A_1A_2'A_3$, then $|r_1| > |r_1'|$ and $-r_1 < -r_1'$ since they both must be taken negative. If P lies inside triangle $A_1A_2'A_3$, we get $-r_1 < 0 < r_1'$. So in either case, using the appropriate sign, we have

$$\pm r_1' + r_2 + r_3 \geq -r_1 + r_2 + r_3.$$

It therefore suffices to prove the extended theorem in the case where all three feet F_i of the distances r_i lie inside the triangle's sides or at its vertices, and when this occurs P lies inside the circumcircle. □

A comprehensive computer run showed the theorem apparently true for all points inside the circumcircle, so all that remained was to prove the theorem when the point P lies outside the triangle and inside the circumcircle. Moreover, Case 2 eliminates a portion of even that region (when, say, P lies inside triangle $A_1A_2'A_3$, for the original Erdős inequality applies to that triangle). Let D be diametrically opposite vertex A_1 on the circumcircle of triangle $A_1A_2A_3$, as shown in Figure 5. Without loss of generality, we must prove the theorem whenever P lies within or on triangle A_2A_3D. We may assume that $A_2 < 90°$ and $A_3 < 90°$ since otherwise the indi-

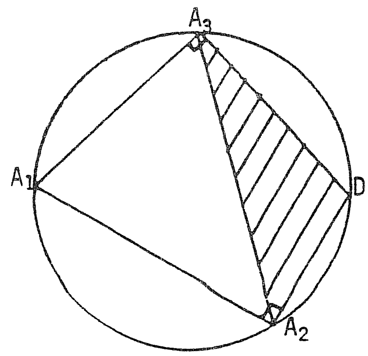

Figure 5

cated region is empty. In this region, since r_1 is to be given a negative sign, we must prove

$$R_1 + R_2 + R_3 + 2r_1 \geq 2r_2 + r_3.$$

If Kazarinoff's statement that his proof holds whenever P lies inside the circumcircle had been substantiated, then the proof of Problem E 2462 would have been complete at this point. The following cases, all developed in the past year, do complete the desired proof.

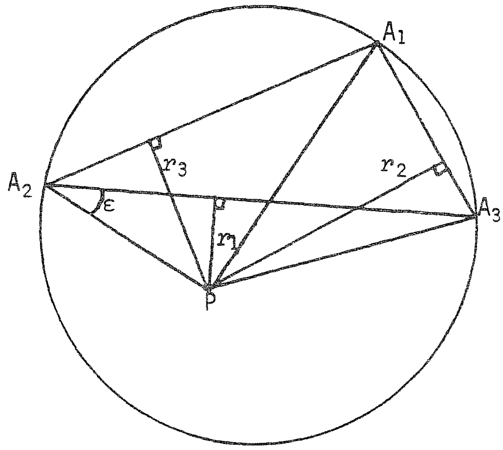

Figure 6

Case 3. Point P lies in triangle A_2A_3D and at least one of angles A_2 and A_3 does not exceed 30°.

Referring to Figure 6, let $A_2 \leq 30^\circ$, so that $\sin A_2 \leq 1/2$. If $\angle A_3A_2P = \varepsilon$, then

$$r_3 = R_2 \sin(A_2+\varepsilon), \qquad r_1 = R_2 \sin\varepsilon,$$

and

$$\begin{aligned}\sin(A_2+\varepsilon) - \sin\varepsilon &= \sin A_2 \cos\varepsilon + \cos A_2 \sin\varepsilon - \sin\varepsilon \\ &= \sin A_2 \cos\varepsilon + \sin\varepsilon(\cos A_2 - 1) \\ &\leq \sin A_2 \leq \tfrac{1}{2}.\end{aligned}$$

Now

$$R_2 = PA_2 > PA_2\{2\sin(A_2+\varepsilon) - 2\sin\varepsilon\} = 2r_3 - 2r_1.$$

Hence

$$R_1 + R_2 + R_3 + 2r_1 \geq r_2 + (2r_3 - 2r_1) + r_2 + 2r_1 = 2r_2 + 2r_3. \quad \square$$

Case 4. Point P lies inside the largest angle of the triangle.

Let P lie in triangle A_2A_3D and suppose $A_1 \geq A_2 \geq A_3$. Then we have $a_1 \geq a_2 \geq a_3$ and $r_1 \leq r_2$, and also

$$2 \leq U \equiv \frac{a_2}{a_3} + \frac{a_3}{a_2} \leq V \equiv \frac{a_3}{a_1} + \frac{a_1}{a_3}.$$

Hence, if for some number N we have $Ur_1 = Vr_2 + N$, then

$$Ur_1 \geq Ur_2 + N$$

and, since also

$$(U-2)r_1 \leq (U-2)r_2,$$

we may subtract to get

$$2r_1 \geq 2r_2 + N.$$

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Therefore, since we have, by the first two lines of (2),

$$R_1 + R_2 + R_3 + \left(\frac{a_2}{a_3}+\frac{a_3}{a_2}\right)r_1 \geq \left(\frac{a_3}{a_1}+\frac{a_1}{a_3}\right)r_2 + \left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)r_3,$$

it follows that

$$R_1 + R_2 + R_3 + 2r_1 \geq 2r_2 + \left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)r_3 \geq 2r_2 + 2r_3. \quad \square$$

Now only one case remains to be settled, but first we prove a pair of lemmas.

LEMMA 1. The function

$$f(x) = 1 - \cos x - \frac{1}{2}\sin(x - 15^\circ)$$

has a minimum at approximately $(29^\circ, 0.044)$, so $f(x) > 0$ for all x in the interval $[15^\circ, 90^\circ]$. (See Figure 7.)

We have

$$f'(x) = \sin x - \frac{1}{2}\cos(x-15^\circ),$$

which vanishes when

$$\tan x = \frac{\cos 15^\circ}{2 - \sin 15^\circ},$$

that is, when $x \approx 29.019466^\circ$, at which point $f(x) \approx 0.004419 > 0$. Since

$$f''(x) = \cos x + \frac{1}{2}\sin(x-15^\circ) > 0,$$

the critical point is a minimum. $\square$

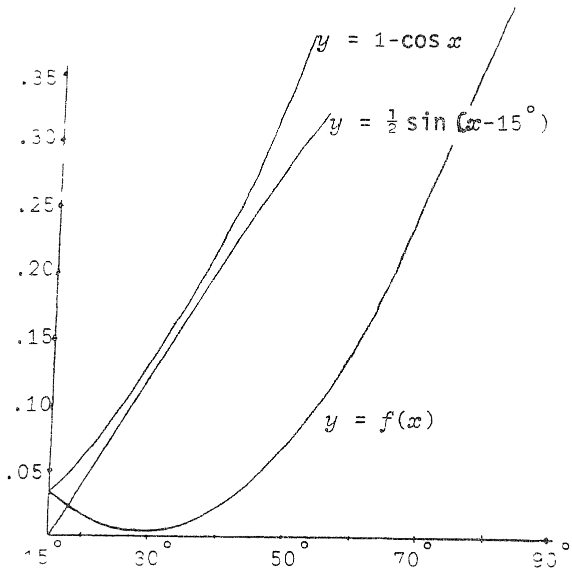

Figure 7

LEMMA 2. If $1 \leq x \leq 2$, then $g(x) = x + 1/x \leq 2.5$.

Clearly $g'(x) \geq 0$ in the given interval, so $g(x) \leq g(2) = 2.5$. $\square$

Our last case takes P inside the triangle A_2A_3D of Figure 5, where $A_2 > 30^\circ$ and $A_3 > 30^\circ$ (by Case 3), and A_1 is not the largest angle of the triangle (by Case 4). We may without loss of generality assume that A_2 is the largest angle. Since we need not consider $A_2 \geq 90^\circ$ (by Figure 5), we have the following case:

Case 5. Point P lies inside triangle A_2A_3D, and $30^\circ < A_3 \leq A_2$ and $A_1 \leq A_2 < 90^\circ$. (See Figure 8.)

Let $\delta = \angle OA_3P$. Since $A_3 > 30^\circ$, we have $A_1 + A_2 < 150^\circ$ and $A_1 < 75^\circ$. So $\angle A_3OA_2 < 150^\circ$ and $\angle OA_3A_2 > 15^\circ$. From

$$\frac{a_2}{a_3} = \frac{\sin A_2}{\sin A_3} \quad \text{and} \quad 30^\circ < A_3 \leq A_2 < 90^\circ,$$

we get

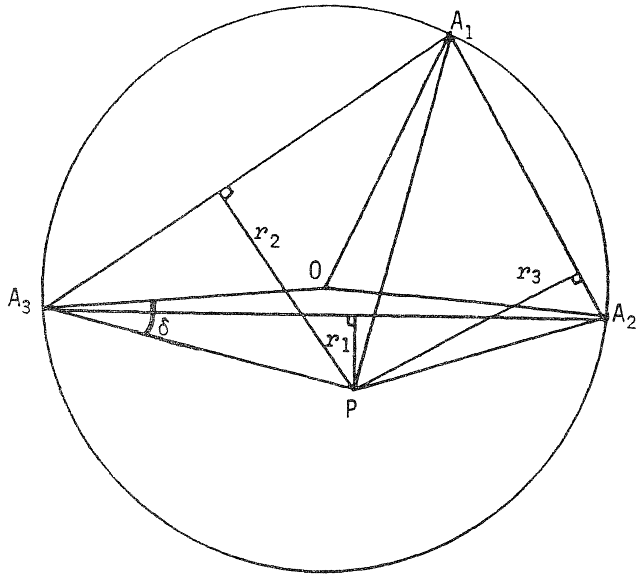

Figure 8

$$1 \le \frac{a_2}{a_3} < \frac{\sin 90^\circ}{\sin 30^\circ} = 2,$$

and so, by Lemma 2,

$$\frac{a_2}{a_3} + \frac{a_3}{a_2} < 2.5.$$

Now $r_1 \le R_3 \sin(\delta - 15^\circ)$, so, by Lemma 1,

$$R_3(1 - \cos\delta) \ge \frac{1}{2}R_3 \sin(\delta - 15^\circ) \ge \frac{1}{2}r_1.$$

Then

$$\begin{aligned} R_1 + R_2 + R_3 + 2r_1 &= R_1 + R_2 + R_3 \cos\delta + R_3(1 - \cos\delta) + 2r_1 \\ &\ge R_1 + R_2 + R_3 \cos\delta + 2.5r_1 \\ &\ge R_1 \cos(OA_1P) + R_2 \cos(OA_2P) + R_3 \cos\delta + \left(\frac{a_2}{a_3} + \frac{a_3}{a_2}\right)r_1 \\ &= \left(\frac{a_3}{a_1} + \frac{a_1}{a_3}\right)r_2 + \left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right)r_3 \qquad \text{by (2)} \\ &\ge 2(r_2 + r_3), \end{aligned}$$

and we are at last finished. The proof of Problem E 2462 is complete.

Finally, we use the extended Erdös-Mordell inequality for triangles to get, as a corollary, a corresponding result for convex quadrilaterals.

Let $A_1A_2A_3A_4$ be a convex quadrilateral, and let P be any point in its plane.

We set $PA_i = R_i > 0$ and denote the signed distance between P and line A_iA_j by r_{ij}, the sign being determined by Demir's convention for any triangle of which A_iA_j is a side. Thus (see Figure 9), r_{12} is associated with triangles $A_1A_2A_3$ and $A_1A_2A_4$ and has the same sign for both triangles regardless of the location of point P; and similar statements can be made about r_{23}, r_{34}, and r_{41}. The distance $|r_{13}|$, on the other hand, is associated with triangles $A_1A_2A_3$ and $A_1A_3A_4$; and if r_{13} is the signed distance associated with triangle $A_1A_2A_3$, then $-r_{13}$ is the signed distance associated with triangle $A_1A_3A_4$. Similarly, if r_{24} corresponds to triangle $A_1A_2A_4$, then $-r_{24}$ corresponds to triangle $A_2A_3A_4$. Our inequality extended to quadrilaterals reads as follows:

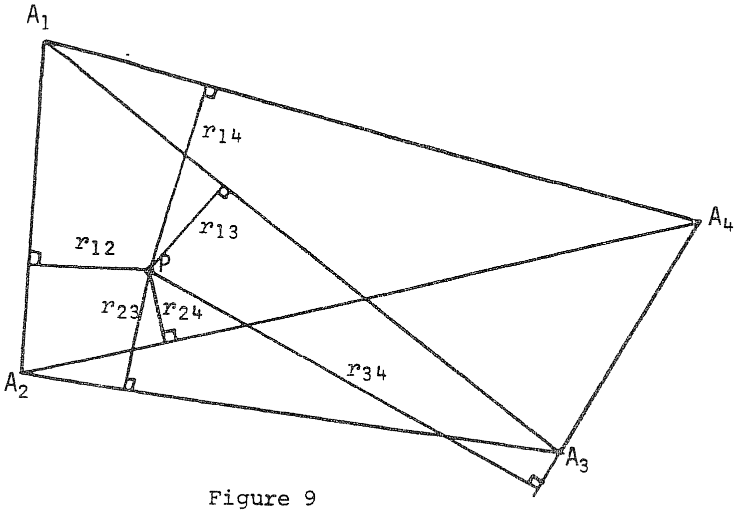

Figure 9

COROLLARY. If $A_1A_2A_3A_4$ is a convex quadrilateral, P is any point in its plane, and the distances R_i and r_{ij} are as defined above, then

$$3(R_1 + R_2 + R_3 + R_4) \geq 4(r_{12} + r_{23} + r_{34} + r_{41}). \tag{3}$$

Proof. We apply the extended Erdös-Mordell inequality successively to triangles $A_1A_2A_3$, $A_1A_2A_4$, $A_1A_3A_4$, and $A_2A_3A_4$:

$$R_1 + R_2 + R_3 \geq 2(r_{12} + r_{23} + r_{13}),$$

$$R_1 + R_2 + R_4 \geq 2(r_{12} + r_{24} + r_{41}),$$

$$R_1 + R_3 + R_4 \geq 2(r_{34} + r_{41} - r_{13}),$$

$$R_2 + R_3 + R_4 \geq 2(r_{23} + r_{34} - r_{24}),$$

and adding these four inequalities yields (3).

REFERENCE

1. D.K. Kazarinoff, "A Simple Proof of the Erdös-Mordell Inequality for Triangles", *Michigan Math. J.*, 4 (1957) 97-98.

Mathematics Department, University of Maine, Orono, Maine 04469.

* * *

Solution to Problem 2625:

American Mathematical Monthly, **85**, (1978), 121.

A Property of Conics

E 2625 [1976, 812]. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

Let A_i $(i \equiv 0, 1, 2, 3 \bmod 4)$ be four points on a circle Γ. Let t_i be the tangent of Γ at A_i and let p_i and q_i be the lines parallel to t_i passing through the points A_{i-1} and A_{i+1} respectively. If $B_i = t_i \cap t_{i+1}$, $C_i = p_i \cap q_{i+1}$ show that the four lines B_iC_i have a point in common.

Solution by Jordi Dou, Barcelona, Spain. We shall prove a more general result.

THEOREM. *Let K be a non-degenerate conic in a real projective plane, A_i $(0 \leqq i \leqq 3)$ be four distinct points on K and r be a line such that $A_i \notin r$. Let t_i be the tangent of K at A_i, $B_i = t_i \cap t_{i+1}$, $T_i = t_i \cap r$, $p_i = T_iA_{i-1}$, $q_i = T_iA_{i+1}$ and $C_i = p_i \cap q_{i+1}$. Then the four lines B_iC_i are concurrent.*

Proof. Let π be the polarity with respect to K and $S = A_0A_2 \cap A_1A_3$. Put $s = \pi(S)$ and $R = \pi(r)$. Let σ be the harmonic homology with center S and axis s. Thus we have $\sigma^2 = 1$ and $\sigma(A_i) = A_{i+2}$. We claim that the point $Q = \sigma(R)$ lies on each of the lines B_iC_i.

Note that π interchanges S and s and consequently σ and π commute. Therefore, $\tau = \sigma\pi = \pi\sigma$ is also a polarity. We have

$$\tau(t_i) = \sigma\pi(t_i) = \sigma(A_i) = A_{i+2},$$

$$\tau(t_{i+1}) = A_{i+3} = A_{i-1},$$

$$\tau(r) = \sigma\pi(r) = \sigma(R) = Q$$

and consequently the two triangles $T_{i+1}T_iB_i$ and $A_{i+2}A_{i-1}Q$ are polar to each other with respect to K. By Chasles' theorem (see H. S. M. Coxeter, *The Real Projective Plane*, Cambridge University Press, 1961, p. 71) this is a pair of Desargues' triangles. Hence the lines $T_{i+1}A_{i+2} = q_{i+1}$, $T_iA_{i-1} = p_i$ and B_iQ are concurrent at $C_i = p_i \cap q_{i+1}$. Therefore we see that Q lies on the lines B_iC_i as claimed.

The statement of the problem is obtained by choosing $K = \Gamma$ and $r =$ line at infinity.

Also solved by L. Kuipers (Switzerland), and the proposer.

Solution to Problem 3135:
American Mathematical Monthly, **95**, (1988), 764.

Matching Distances to Vertices

E 3135 [1986, 215]. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

For a scalene triangle ABC inscribed in a circle, prove that there is a point D on the circle whose distance from the opposite vertex is the sum of its distances from the other two vertices, and construct D with ruler and compass.

Solution I by J. Leech, University of Stirling, Scotland. Suppose $BC > AC > AB$. Perturb BAC into an isosceles triangle BXY by determining X on the ray BA and Y on BC such that $BX = BY = AC$. The circumcircle of BXY meets the circumcircle of BAC at the required point D. To prove D has the required property, extend AD to a point Z such that $DZ = DC$. Then triangle DZC is similar to triangle BXY, because the angles at D and B are both supplementary to angle ADC. Consequently, triangle DBY is congruent to triangle ZAC, since $AC = BY$ by construction, the angles at A and B are equal and angle BDY = angle BXY = angle DZC. Hence $DB = ZA = DA + DC$.

Solution II by P. Tzermias (student), University of Patras, Greece. Let a, b, c be the lengths of the sides opposite A, B, C, and let x, y, z be the lengths of DA, DB, DC. We seek point D such that $y = x + z$. By Ptolemy's Theorem, $ax + cz = by$. Substituting for y yields $x/z = (c - b)/(b - a)$. Thus, D lies on the "Circle of Apollonius" determined by A and C using the ratio $(c - b)/(b - a)$. This circle has a standard construction (see N. Altshiller-Court, *College Geometry*, 1952, p. 15).

Editorial comments. Since $c - b$ and $b - a$ must have the same sign, D must lie on the arc cut by the side of intermediate length. Consequently, if ABC is isosceles, then D can only be the vertex common to the two equal sides. On the other hand, if ABC is equilateral, then D can be any point on the circumference; Leech's two circles then coincide.

Several solvers noted that the existence of D follows from the Intermediate Value Theorem. If D on the arc opposite B is close to the shortest side of ABC, then its distance to B is less than $DA + DC$, but if D is close to the longest side, then $DB > DA + DC$.

Other solutions independent of Ptolemy's Theorem were submitted by J. Dou (Spain), L. Kuipers (Switzerland), and by P. L. Hon (Hong Kong).

E. Morgantini (Italy) submitted a paper entitled "Una Quartica Bicircolare Della Geometria Del Triangolo" making reference to this problem.

In addition to the solvers mentioned above, correct solutions were received from S. Arslanagić (Yugoslavia), A. Bager, H. Eves, J. Fukuta (Japan), H. Kappus (W. Germany), O. P. Lossers (Netherlands), J. P. Robertson, J. S. Robertson, V. Schindler (E. Germany), R. A. Simon (Chile), B. A. Troesch, M. Vowe (Switzerland), and the proposer.

Solution to Problem 3164:
American Mathematical Monthly, **95**, (1988), 660.

Elliptical Tangents

E 3164 [1986, 566]. *Proposed by Huseyn Demir, Middle East Technical University, Ankara, Turkey.*

Let s, t be the lengths of the tangent line segments to an ellipse from an exterior point. Find the extreme values of the ratio s/t.

Solution by Gene Arnold and Vaclav Konecny, Ferris State College, Big Rapids, MI. Consider an ellipse as the normal projection of a circle, from one plane to another in $\mathbf{R}^3$. Clearly tangents project to tangents and the minor axis of the ellipse is perpendicular to the intersection of the two planes. Since the ratio of any two intersecting tangents to the circle is 1, the extreme ratios of two such tangents to the ellipse will be attained when the ratio of one tangent to the circle to its projection is minimum while the other ratio is maximum. This happens when the projected tangents are respectively parallel to and perpendicular to the intersection of the planes. Thus the extreme ratios are those of the major axis to the minor axis, and its reciprocal.

Editorial comment. M. S. Klamkin suggested study of the more difficult problem of the extreme values of $|s - t|$.

Also solved by M. Barr (Canada), J. C. Binz (Switzerland), J. M. Cohen, J. Dou (Spain), J. Fukuta (Japan), P. L. Hon (Hong Kong), L. R. King, M. S. Klamkin (Canada), K.-W. Lau (Hong Kong), O. P. Lossers (The Netherlands), M. Pachter (South Africa), K. Schilling, J. H. Steelman, P. Tracy, D. B. Tyler, C. Vandermee (The Netherlands), and the proposer. One incorrect solution was received.

Solution to Problem 3422:
American Mathematical Monthly, **99**, (1992), 679.

Tangents Intersect on the Axis of Involution

E 3422 [1991, 158]. *Proposed by H. Demir and C. Tezer, Middle East Technical University, Ankara, Turkey.*

Suppose F and F' are points situated symmetrically with respect to the center of a given circle, and suppose S is a point on the circle not on the line FF'. Let P and P' be the second points of intersection of SF and SF' respectively with the circle. If the tangents to the circle at P and P' intersect at T, prove that the perpendicular bisector of FF' passes through the midpoint of the line segment ST.

Solution I by Jean-Pierre Grivaux, Paris, France. We work in the complex plane, with lower-case letters denoting the complex representations of points designated by the corresponding upper-case letters. We may assume that the circle is $\mathbf{U} = \{Z\colon |z| = 1\}$ and that the points F and F' are on the real axis.

If $A, B \in \mathbf{U}$, then Z is on the line through A and B if and only if $z + ab\bar{z} = a + b$, which we shall refer to as equation $\mathscr{E}_{ab}$. To derive this equation, note that the line is the set of Z whose numerical representation satisfies $z = a + r(b - a)$, where r is real. Conjugating this and using $\bar{a} = 1/a$ and $\bar{b} = 1/b$ yields $\bar{z} = \bar{a} + r(\bar{b} - \bar{a})$, which when multiplied by ab and added to the first equation yields $\mathscr{E}_{ab}$. This form of $\mathscr{E}_{ab}$ remains valid when $a = b$.

Since $\mathscr{E}_{pp}$ and $\mathscr{E}_{p'p'}$ are the equations of the tangents to $\mathbf{U}$ at P and P', we have $t + p^2\bar{t} = 2p$ and $t + (p')^2\bar{t} = 2p'$. Solving for t by eliminating $\bar{t}$ (when $p \neq p'$) yields $t = 2/(\bar{p} + \bar{p'})$. Note that $p + p' \neq 0$ because s is not real. The midpoint of ST is Z, where

$$z = \frac{1}{2}(s + t) = \frac{1}{2}\left(s + \frac{2}{\bar{p} + \bar{p'}}\right),$$

and the result we want to prove is $z + \bar{z} = 0$, which by the above is

$$\left(s + \frac{2}{1/p + 1/p'}\right) + \left(\frac{1}{s} + \frac{2}{p + p'}\right) = 0.$$

This is equivalent by algebraic manipulation to

$$\left(-\frac{2s}{1 + s^2}\right)(1 + pp') = p + p'. \qquad (*)$$

Sine F and F' belong to the lines PS and $P'S$ respectively, f and $f'(= -f)$ satisfy the equations $\mathscr{E}_{ps}$ and $\mathscr{E}_{p's}$ respectively, namely $(f)(1 + ps) = p + s$ and $(-f)(1 + p') = p' + s$, where we use the fact that $\bar{f} = f$. Elimination of f from these two equations produces the desired equality $(*)$.

Solution II by the proposers. We exclude the case in which F and F' coincide. Let K be the point diametrically opposite S. Let S' be the additional point where the line through S parallel to FF' intersects the circle (S' may coincide with S). The lines SS', SP, SK, SP' form a harmonic pencil, as the center of the circle bisects FF'. Consequently, for any point X on the circle, the lines XS', XP, XK, XP' form a harmonic pencil. Choosing $X = P$ or $X = P'$ in particular, we find that the pencils PS', PT, PK, PP' and $P'S', P'P, P'K, P'T$ are harmonic. Since the line PP' is common to both pencils, the points S', K, T lie on a line which is clearly perpendicular to FF'. Hence the perpendicular bisector of FF' bisects ST.

Editorial comment. Most solvers used straightforward analytic geometry and brute force calculation to prove the result. Several used synthetic Euclidean geometry. H. Kappus gave another proof using complex numbers. O. P. Lossers gave another proof using projective geometry. A nice approach by J. Dou uses a classical property of projective involutions of a conic (involutions sending a conic to itself and preserving cross ratios). We briefly describe this and its relationship to Grivaux's solution, using the notational conventions of that solution.

The mapping $\sigma: \mathbb{R} \to \mathbb{R}$ given by $\sigma(x) = -x$ yields a projective involution of the real line which extends to a projective involution of the real projective line $\mathbf{P}$ by defining $\sigma(\infty) = \infty$. With point S given on the unit circle $\mathbf{U}$, we define $\pi: \mathbf{P} \to \mathbf{U}$ by letting $\pi(X)$ be the point where the line joining S to $X \in \mathbf{P}$ again intersects $\mathbf{U}$. In particular, π applied to the point at infinity is the other point of intersection with $\mathbf{U}$ of the line through S parallel to the real axis. It then follows that the mapping $g: \mathbf{U} \to \mathbf{U}$ given by $g = \pi \circ \sigma \circ \pi^{-1}$ is a projective involution of $\mathbf{U}$.

The numbers corresponding to the fixed points of g are $-s$ and $-\bar{s}$, since σ fixes 0 and ∞. Now a classical result of projective geometry implies that for each $P \in \mathbf{U}$, the tangent lines of $\mathbf{U}$ at P and $g(P)$ intersect on the line $\mathbf{l}$ through the fixed points of g. Since T is the intersection of the tangent lines at P and $P' = g(P)$, we see that T is on the line $\mathbf{l}$, and it is then obvious that the midpoint of ST is on the pure-imaginary axis, as was to be proved.

It is easy to calculate that $\pi(X)$ is represented by $(x - s)/(1 - xs)$ for $x \in \mathbb{R}$, and $g(P)$ is represented by $(\lambda - p)/(1 - \lambda p)$ for $p \in \mathbf{U}$, where $\lambda = -2s/(1 + s^2)$. In fact, λ is real (or ∞) and it represents the intersection of the tangent lines at the fixed points of g. Indeed, the relation $(*)$ in Grivaux's solution expresses the fact that λ satisfies the equation $\mathscr{E}_{pp'}$; thus the line through P and P' always passes through Λ. This shows that the involution g is obtained by sending each point $P \in \mathbf{U}$ to the other point where $\mathbf{U}$ intersects the line through Λ and P. The line $\mathbf{l}$ (the "axis" of the involution g) is the polar of Λ with respect to $\mathbf{U}$.

For a detailed discussion of involutions of conics, see H. F. Baker, *An Introduction to Plane Geometry* (Cambridge University Press, 1943), Chapter IX, or M. Berger, *Geometry II* (Springer, 1987), Section 16.3. In Berger's book the above point λ is called the "Frégier point" of the involution g.

Solved by 26 readers (including those cited) and the proposers.

Solution to Problem 3469:
American Mathematical Monthly, **100**, (1993), 875.

Six Barycenters in Search of a Conic

E3469 [1991, 955]. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

Suppose P is a point in the interior of triangle ABC and suppose AP, BP, CP meet the lines BC, CA, AB respectively at the points D, E, F. Prove that the centroids of the six triangles $PBD, PDC, PCE, PEA, PAF, PFB$ lie on a conic if and only if P lies on at least one of the three medians of the triangle.

Restatement of problem and fixing of notation. Applying the homothety with center P and ratio $3:2$ we see that the centroids of triangles are on a conic if and only if the midpoints of AF, FB, BD, DC, CE and EA are on one conic. Let x, y, z, u, v, w denote half the lengths of AF, FB, BD, DC, CE, EA, respectively. Let the midpoints of AF, FB, BD, DC, CE, EA be denoted by $1, 2, 4, 5, 6$ respectively.

Solution I by Victor Prasolov, Independent University of Moscow, Moscow, Russia.

By Carnot's Theorem (see Howard W. Eves, *A survey of geometry* (Revised Edition), Allyn and Bacon, 1972, pages 256 and 262) the six centroids lie on a conic if and only if

$$x(2x+y)z(2z+u)v(2v+w) = w(2w+v)u(2u+z)y(2y+x). \qquad (1)$$

By Ceva's Theorem, $xzv = wuy$, so (1) simplifies to $xzw + zvy + vxu - (wux + uyv + ywz) = 0$, or $(x-y)(z-u)(w-v) = 0$. This condition corresponds to P lying on a median.

Solution II by Albert Nijenhuis, Seattle, WA. By Pascal's Theorem, the points 1, 2, 3, 4, 5, and 6 lie on a conic if and only if the three points $Q = AB \cap 45$, $R = BC \cap 61$ and $S = CA \cap 23$ are collinear. (There is no real difficulty if any of these points are at infinity. The ratio AQ/QB, for example, is replaced by -1 if $AB \| 45$.)

By Menelaus' Theorem, we have

$$\frac{AQ}{QB} \cdot \frac{2z+u}{u} \cdot \frac{v}{2w+v} = -1, \qquad \frac{BR}{RC} \cdot \frac{2v+w}{w} \cdot \frac{x}{2y+x} = -1,$$

$$\frac{CS}{SA} \cdot \frac{2x+y}{y} \cdot \frac{z}{2u+z} = -1.$$

Multiplying these together and using Ceva's theorem, as in Solution I, we see that $AQ/QB \cdot BR/RC \cdot CS/SA = -1$ if and only if $(x-y)(z-u)(w-v) = 0$. Thus Q, R, S are collinear and hence the points $1, 2, 3, 4, 5, 6$ lie on a conic if and only if P is on a median.

Comments by Neela Lakshmanan, University of Scranton, Scranton, PA. The restriction that P is interior to the triangle may be relaxed: we need only that P does not lie on any side of the triangle.

We can prove that the result is true not only for the midpoints but also for the points that divide each of those six segments in a *constant ratio*: If $1, 2, 3, 4, 5, 6$ are points on the sides of the triangle defined by $A1:1F = F2:2B = B3:3D = D4:4C = C5:5E = E6:6A$, then the six points lie on a conic if and only if P is on a median. Also, if P is an interior point, the hexagon $1, 2, 3, 4, 5, 6$ is convex and attains its maximum area when P is the centroid of $\triangle ABC$.

Editorial comment. Many of the solvers supplemented the use of Carnot's Theorem or Pascal's Theorem with homogeneous coordinates and analytic methods. Some others worked directly with conditions on the six coefficients of a general conic.

Solved also by F. Bellot and M. A. Lopéz (Spain), R. J. Chapman (U.K.), J. Fukuta (Japan), H. Kappus (Switzerland), O. P. Lossers (The Netherlands), I. A. Sakmar (Turkey), Anchorage Math Solutions Group, and the proposer. One incorrect solution was received.

5 Contributed Solutions to *MONTHLY* problems

List of solutions contributed by Hüseyin Demir to problems in American Mathematical Monthly:

[1] Advanced Problem 4057, *American Mathematical Monthly*, 51, (1944), 168.
[2] Elementary Problem 1107, *American Mathematical Monthly*, 61, (1954), 643.
[3] Elementary Problem 1142, *American Mathematical Monthly*, 62, (1955), 444.
[4] Elementary Problem 1148, *American Mathematical Monthly*, 62, (1955), 495.
[5] Elementary Problem 1166, *American Mathematical Monthly*, 63, (1956), 42.
[6] Elementary Problem 1687, *American Mathematical Monthly*, 72, (1965), 425.
[7] Elementary Problem 2122, *American Mathematical Monthly*, 76, (1969), 833.
[8] Elementary Problem 2398, *American Mathematical Monthly*, 81, (1974), 89.

Solution to Problem 4057:
American Mathematical Monthly, 51, (1944), 168.

Euler Line

4057 [1942, 616]. *Proposed by J. R. Musselman, Western Reserve University*

Let B_1, B_2, B_3 be the points symmetric to the vertices of triangle $A_1A_2A_3$ in its circumcenter O, and let C_1, C_2, C_3 be the reflections of A_i in the perpendicular bisector of the sides of $A_1A_2A_3$. It is known that the circles OB_1C_1, OB_2C_2, OB_3C_3 meet at a point P. Show that P lies on the Euler line of $A_1A_2A_3$ and that O is the midpoint of PD, where D is the inverse in the circumcircle of the orthocenter H of $A_1A_2A_3$.

Solution by Hüseyin Demir, Columbia University. Let $G_1G_2G_3$ be the triangle formed by the straight lines A_iC_i so that $A_1A_2A_3$ is its medial triangle, the circumcircle (O) of the latter is its ninepoint circle, G_iC_i are its altitudes, its orthocenter H' is the symmetric of H with respect to O. and the straight lines C_iB_i are concurrent in H'. Let P be the point where the circle (OB_1C_1) cuts OH', *i.e.*, OH. We have $H'O\cdot H'P=H'C_1\cdot H'B_1=H'C_1\cdot H'B_i$; hence the circles (OB_iC_i) intersect again in P. The inverse of (OB_1C_1) with respect to (O) is B_1C_1, and hence $OH'\cdot OP=\overline{OC}_1{}^2=R^2$. Since $OH\cdot OD=R^2$ and $OH=H'O$, we must have $OD=PO$.

Solved also by H. Eves using inversion with respect to O and power $-R^2$ which gives a concise proof.

Solution to Problem 1107:
American Mathematical Monthly, 61, (1954), 643.

A Pencil of Planes Associated with a Tetrahedron

E 1107 [1954, 194]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

On the edges AB, AC, AD of a tetrahedron $ABCD$ are marked points M, N, P such that $AB=nAM$, $AC=(n+1)AN$, $AD=(n+2)AP$. Show that the plane MNP contains a fixed line as n varies.

I. *Solution by Hüseyin Demir, Zonguldak, Turkey.* From the relations it is evident that the ranges of points $[M]$ and $[P]$ are projective. But since A is a self-corresponding element, the projectivity is a perspectivity. Hence MP is on a fixed point P'. Similarly MN is on a fixed point N'. Hence the plane MNP is on the fixed line $P'N'$.

Solution to Problem 1142:
American Mathematical Monthly, 62, (1955), 444.

Semi-vertical Angle of a Right Circular Cone

E 1142 [1954, 711]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

Find the semi-vertical angle of a right circular cone if three generating lines make angles of 2α, 2β, 2γ, with each other.

Demir gave the equivalent answer

$$\sin^2\phi = -\frac{16\sin^2\alpha\sin^2\beta\sin^2\gamma}{\begin{vmatrix} 0 & \sin\alpha & \sin\beta & \sin\gamma \\ \sin\alpha & 0 & \sin\gamma & \sin\beta \\ \sin\beta & \sin\gamma & 0 & \sin\alpha \\ \sin\gamma & \sin\beta & \sin\alpha & 0 \end{vmatrix}}.$$

Solution to Problem 1148:
American Mathematical Monthly, 62, (1955), 495.

Two Equiareal Triangles

E 1148 [1955, 40]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let a, b, c be arbitrary points on the sides BC, CA, AB of triangle ABC, and let A', B', C' be the reflections of A, B, C in the midpoints of the segments bc, ca, ab. Show that triangles abc and $A'B'C'$ have equal areas.

Solution by Hüseyin Demir, Zonguldak, Turkey. Let a', b', c' be the reflections of a, b, c in the midpoints of BC, CA, AB. Since, by a well known property, abc and $a'b'c'$ have equal areas, we shall prove that $a'b'c'$ and $A'B'C'$ have equal areas. From $\overrightarrow{aB'}=\overrightarrow{Bc}=\overrightarrow{c'A}$, $\overrightarrow{aC'}=\overrightarrow{Cb}=\overrightarrow{b'A}$ we get $b'c'=B'C'$. Similarly $c'a'=C'A'$, $a'b'=A'B'$, and triangles $a'b'c'$ and $A'B'C'$ are actually congruent.

Also solved by W. B. Carver, A. R. Hyde, M. S. Klamkin, D. C. B. Marsh, C. S. Ogilvy, C. F. Pinzka, Roscoe Woods, and the proposer.

Pinzka called attention to two similar results in R. A. Johnson, *Modern Geometry* (1929), p. 80. Carver, Hyde, Ogilvy, and Woods gave simple solutions using oblique coordinates.

Editorial Note. The above solution shows that triangles $a'b'c'$, $A'B'C'$ are not only congruent, but also homothetic. It follows that if a, b, c are collinear on a line L, then A', B', C' are also collinear on a line L' parallel to the reciprocal transversal of L. Consequently, if L is a Simson line of triangle ABC, then L and L' are perpendicular.

Solution to Problem 1166:
American Mathematical Monthly, 63, (1956), 42.

Chain of Circles in a Segment

E 1166 [1955, 364]. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Let DE be a variable chord perpendicular to diameter AB of a given circle (O). The maximum circle (ω_0) inscribed in the smaller segment, DEB, touches chord DE in C. The circle (ω_1) is tangent to (ω_0), (O), and DC and another circle (ω_2) is tangent to (ω_1), (O), and DC. Find the ratio BC/CA for which the radius of circle (ω_2) is a maximum.

Solution by Hüseyin Demir, Zonguldak, Turkey. Denote the radii of (O) and (ω_i) by R and r_i respectively. Let (ω_1), (ω_2) touch CD in C_1, C_2. Then we easily get

$$CC_1 = 2\sqrt{r_0 r_1}, \qquad C_1C_2 = 2\sqrt{r_1 r_2}.$$

From right triangles having hypotenuses $O\omega_1 = R - r_1$, $O\omega_2 = R - r_2$ we get

$$(R - 2r_0 + r_1)^2 + 4r_0r_1 = (R - r_1)^2, \tag{1}$$

$$(R - 2r_0 + r_2)^2 + 4(\sqrt{r_0r_1} + \sqrt{r_1r_2})^2 = (R - r_2)^2. \tag{2}$$

The value

$$r_1 = r_0(R - r_0)/R,$$

obtained from (1), when substituted in (2) yields

$$r_2 = (R - r_0)^2 r_0/(R + r_0)^2.$$

Now, introducing $k = BC/CA = r_0/(R - r_0)$ and applying the derivative test for a maximum, we get

$$k = (\sqrt{5} - 1)/4.$$

Also solved by G. B. Charlesworth, Walter Guber, A. R. Hyde, R. B. Plymale, and the Proposer. Some of these solutions were based upon a misinterpretation of the figure of the problem.

The Proposer remarked that the problem was suggested by an attempt to display circle (ω_2) to best advantage in a diagram. The following interesting allied facts were pointed out by the Proposer:

1. Circles (ω_0) and (ω_1) are maximum when C coincides with O, but (ω_2) is a maximum when $BC/CA = (\sqrt{5}-1)/4$, with the unexpected consequence that *CB is the side of a regular decagon inscribed in the circle on AC as diameter.*
2. $r_{2(\max)} = r_1/2$.
3. $r_n = 2AB \cos^2 u/[\tan^n (u/2) + \cot^n (u/2)]^2$, u being the angle ABD, (communicated to the Proposer by Victor Thébault).
4. r_n is rational if AC and CB are rational.

Solution to Problem 1687:
***American Mathematical Monthly*, 72, (1965), 425.**

An Application of Menelaus' Theorem

E 1687 [1964, 430]. *Proposed by Daniel Pedoe, Purdue University*

UVW is an equilateral triangle; A, B, C are the respective midpoints of the sides VW, WU, UV; A' is any point on line VW, B' any point on line WU, and C' any point on line UV. If P is the intersection of BC and $B'C'$, Q of CA and $C'A'$, R of AB and $A'B'$, prove that (1) the lines $A'P$, $B'Q$, $C'R$ are concurrent, (2) the areal coordinates of the point of concurrency with respect to triangle ABC are, with a suitable sign convention, $(AA')^{-1}:(BB')^{-1}:(CC')^{-1}$.

Generalize both (1) and (2) by means of an affine projection, and generalize (1) by a general projection.

I. *Solution by Huseyin Demir, Middle East Technical University, Ankara, Turkey.* (1) We first set $BC=CA=AB=1$ and consider VW, WU, UV; $AA'=a'$, $BB'=b'$, $CC'=c'$ as directed segments. Let λ, μ, ν be the ratios in which P, Q, R divide the sides of $A'B'C'$. Applying the Menelaus theorem to the pair $UC'B'$, CB we get $(PB'/PC')\cdot(CC'/CU)\cdot(BU/BB')=1$ or $\lambda(-c')(1/b')=1$; i.e., $\lambda=-b'/c'$. Considering also two other pairs we get $\mu=-c'/a'$ and $\nu=-a'/b'$ which give $\lambda\cdot\mu\cdot\nu=-1$ proving the concurrency at a point T.

(2) We denote the areal coordinates of T by the matrices $(l'm'n')$ and $(l\ m\ n)$ in the triangles $A'B'C'$ and ABC respectively, and from $l':n'=-\mu$, $m':l'=-\nu$ we obtain

$$(\alpha)\qquad l':m':n' = l':\frac{a'}{b'}\ l':\frac{a'}{c'}\ l' = \frac{1}{a'}:\frac{1}{b'}:\frac{1}{c'}\ .$$

Now to find $l:m:n=(l\ m\ n)$, let us first introduce the following symbol

$$LMN/XYZ = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$$

where the columns are the areal coordinates of L, M, N in the triangle XYZ. We have

$$(\beta)\quad A'B'C'/UVW = \begin{bmatrix} 0 & 1-a' & 1+a' \\ 1+b' & 0 & 1-b' \\ 1-c' & 1+c' & 0 \end{bmatrix},\quad UVW/ABC = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix},$$

and $T/A'B'C'=(1/a'\ 1/b'\ 1/c')$. It is not difficult to see the general identity

$$(\gamma)\qquad T/ABC = (T/A'B'C')\cdot(A'B'C'/UVW)\cdot(UVW/ABC).$$

Substituting (α) and (β) in (γ),

$$(l\ m\ n) = (l'\ m'\ n')\cdot\begin{bmatrix} 0 & 1-a' & 1-a' \\ 1-b' & 0 & 1-b' \\ 1-c' & 1-c' & 0 \end{bmatrix}\cdot\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= (1/a'\ 1/b'\ 1/c')\begin{bmatrix} 2 & 2a' & -2a' \\ -2b' & 2 & 2b' \\ 2c' & -2c' & 2 \end{bmatrix} = (2/a'\ 2/b'\ 2/c')$$

$$(\delta) \qquad T/ABC = l:m:n = 1/a':1/b':1/c' = AA'^{-1}:BB'^{-1}:CC'^{-1}.$$

(3) An affine projection transforms UVW into an arbitrary triangle and ABC into its medial triangle in which the concurrency holds.

In an affine transformation the ratio of segments and the ratio of areas being preserved, replacing $AA'/1$, $BB'/1$, $CC'/1$ by AA'/VW, BB'/WU, CC'/UV, (δ) becomes

$$T/ABC = \left(\frac{AA'}{VW}\right)^{-1}:\left(\frac{BB'}{WU}\right)^{-1}:\left(\frac{CC'}{UV}\right)^{-1}.$$

By a general projection the perspective triangles UVW and ABC are transformed into such triangles. So the generalization is obtained for any UVW and for any inscribable triangles ABC and $A'B'C'$, such that AU, BV, CW are concurrent. Furthermore, the point P has the same areal coordinates in ABC and $A'B'C'$.

Also solved by the proposer who points out that the generalization of (1) to general projections occurs as a problem in a 1909 Mathematical Tripos, Part I.

Solution to Problem 2122:
American Mathematical Monthly, 76, (1969), 833.

An Extension of Napoleon's Theorem

E 2122 [1968, 898]. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

Let D, E and F be points in the plane of a nonequilateral triangle ABC so that triangles BDC, CEA and AFB are directly similar. Prove that triangle DEF is equilateral if and only if the three triangles are isosceles (with a side of triangle ABC as base) with base angles 30°. (The "if" part, Napoleon's theorem, is known. See the MATHEMATICS MAGAZINE, 1966, p. 166.)

Solution by Huseyin Demir, Middle East Technical University, Ankara, Turkey. The following lemma is easily proved:

LEMMA. A triangle in the complex plane with vertices a, b and c is equilateral if and only if $a^2+b^2+c^2-bc-ca-ab=0$.

Let a, b, c; d, e, f be the affixes of the vertices of the triangles ABC, DEF. Since the triangles DBC, ECA, FAB are directly similar, then for some t

$$d = b + (c - b)t, \qquad e = c + (a - c)t, \qquad f = a + (b - a)t.$$

Forming the expression $U=d^2+e^2+f^2-ef-fd-de$, we find

$$U = (a^2 + b^2 + c^2 - bc - ca - ab)(3t^2 - 3t + 1).$$

If ABC is equilateral, then by the lemma, $U=0$, and again by lemma, DEF is equilateral. Now suppose that DEF is equilateral, that is $U=0$ by lemma. Since ABC is supposed to be non-equilateral, we must have $3t^2-3t+1=0$. Solving for t, we find $t=\frac{1}{3}\sqrt{3}$ cis $(\pm\pi/6)$ which proves the assertion.

Also solved by Walter Bluger, Slobodan Ćuk (Yugoslavia), M. G. Greening (Australia). L. Kuipers, C. F. Merrill, and the proposer. Jordi Dou (Spain) shows the uniqueness of the solution. A. W. Walker mentions a weaker result given in a paper by Wong, this MONTHLY, 48 (1941), p. 530.

Solution to Problem 2398:
American Mathematical Monthly, 81, (1974), 89.

A Result Known to Johnson

E 2398 [1973, 202]. *Proposed by C. W. Dodge, University of Maine at Orono*

Prove that the point of intersection of the diagonals of a parallelogram lies on the pedal circle for any vertex with respect to the triangle formed by the other three vertices.

I. *Solution by Huseyin Demir, Middle East Technical University, Ankara, Turkey.* Let $ABCD$ be a given parallelogram with I as center. Let the projections of D on sides BC, CA, AB of triangle ABC be A', B', C', respectively. If $\measuredangle D = \pi/2$, the pedal triangle degenerates into the Simson line AC containing the point I.

We give the proof in the case where $\measuredangle D > \pi/2$ and A' is on the segment BC and C' is on the segment AB. Similar proofs may be given in other cases. We need only show that $\measuredangle C'A'I = \measuredangle AB'C'$. In obtaining this equality we use the properties that $AC'B'D$ and $DC'BA'$ are cyclic and triangle DIA' is isosceles. We have

$$\begin{aligned}\measuredangle C'A'I &= \measuredangle C'A'D - \measuredangle IA'D = \measuredangle C'BD - \measuredangle IDA' = \measuredangle A'DC \\ &= \pi/2 - \measuredangle C = \pi/2 - \measuredangle A = \measuredangle ADC' = \measuredangle AB'C'.\end{aligned}$$

II. *Solution by A. W. Walker, Toronto, Canada.* Let D be the reflection of the vertex A of triangle ABC in the midpoint M of the side BC. If BAC is a right triangle, the pedal "circle" of D for triangle ABC is the line BC; if not, let E be the meet of the lines tangent to circle ABC at B and C. Then BD and BE are isogonal conjugate lines in the angle ABC, and likewise for CD and CE in angle BCA, so D and E are isogonal conjugate points in triangle ABC and therefore (R. A. Johnson, *Modern Geometry*, p. 155) have a common pedal circle passing through the projection M of E on BC.

REMARK. E 2398 is a special case of the theorem: *For a plane non-orthocentric quadrangle ABCD there are four pedal circles (and four nine-point circles) passing through the center of the rectangular hyperbola ABCD* (Johnson, p. 242).

Also solved by Günter Bach (Germany), Leon Bankoff, Howard Eves, Michael Goldberg, M. G. Greening (Australia), Lew Kowarski, L. Kuipers, and the proposer.

6 Proposals for *Mathematics Magazine*

List of Proposals composed by Hüseyin Demir

[1] Proposal 208, *Mathematics Magazine*, 28, (1954-1955), 27.
[2] Proposal 217, *Mathematics Magazine*, 28, (1954-1955), 103.
[3] Proposal 227, *Mathematics Magazine*, 28, (1954-1955), 160.
[4] Proposal 234, *Mathematics Magazine*, 28, (1954-1955), 234.
[5] Proposal 242, *Mathematics Magazine*, 28, (1954-1955), 284.
[6] Proposal 248, *Mathematics Magazine*, 29, (1955-1956), 46.
[7] Proposal 258, *Mathematics Magazine*, 29, (1955-1956), 163.
[8] Proposal 266, *Mathematics Magazine*, 29, (1955-1956), 222.
[9] Proposal 298, *Mathematics Magazine*, 30, (1956-1957), 164.
[10] Proposal 304, *Mathematics Magazine*, 30, (1956-1957), 223.
[11] Proposal 334, *Mathematics Magazine*, 31, (1957-1958), 228.
[12] Proposal 349, *Mathematics Magazine*, 32, (1958-1959), 47.
[13] Proposal 372, *Mathematics Magazine*, 32, (1958-1959), 220.
[14] Proposal 380, *Mathematics Magazine*, 32, (1958-1959), 278.
[15] Proposal 384, *Mathematics Magazine*, 33, (1969-1960), 51.
[16] Proposal 398, *Mathematics Magazine*, 33, (1969-1960), 165.
[17] Proposal 407, *Mathematics Magazine*, 33, (1959-1960), 225.
[18] Proposal 415, *Mathematics Magazine*, 33, (1959-1960), 296.
[19] Proposal 419, *Mathematics Magazine*, 34, (1960-1961), 49.
[20] Proposal 425, *Mathematics Magazine*, 34, (1960-1961), 109.
[21] Proposal 437, *Mathematics Magazine*, 34, (1961), 174.
[22] Proposal 440, *Mathematics Magazine*, 34, (1961), 237.
[23] Proposal 458, *Mathematics Magazine*, 34, (1961), 364.
[24] Proposal 472, *Mathematics Magazine*, 35, (1962), 55.
[25] Proposal 487, *Mathematics Magazine*, 35, (1962), 186.
[26] Proposal 498, *Mathematics Magazine*, 35, (1962), 309.
[27] Proposal 509, *Mathematics Magazine*, 36, (1963), 133.
[28] Proposal 517, *Mathematics Magazine*, 36, (1963), 197.
[29] Proposal 529, *Mathematics Magazine*, 36, (1963), 264.
[30] Proposal 537, *Mathematics Magazine*, 37, (1964), 55.
[31] Proposal 544, *Mathematics Magazine*, 37, (1964), 119.
[32] Proposal 563, *Mathematics Magazine*, 37, (1964), 276.
[33] Proposal 572, *Mathematics Magazine*, 38, (1965), 52.
[34] Proposal 587, *Mathematics Magazine*, 38, (1965), 179.
[35] Proposal 599, *Mathematics Magazine*, 38, (1965), 241.

[36] Proposal 600, *Mathematics Magazine*, 38, (1965), 317.
[37] Proposal 609, *Mathematics Magazine*, 39, (1966), 69.
[38] Proposal 628, *Mathematics Magazine*, 39, (1966), 246.
[39] Proposal 639, *Mathematics Magazine*, 39, (1966), 306.
[40] Proposal 649, *Mathematics Magazine*, 40, (1967), 100.
[41] Proposal 680, *Mathematics Magazine*, 41, (1968), 42.
[42] Proposal 724, *Mathematics Magazine*, 42, (1969), 96.
[43] Proposal 738, *Mathematics Magazine*, 42, (1969), 214.
[44] Proposal 743, *Mathematics Magazine*, 42, (1969), 267.
[45] Proposal 756, *Mathematics Magazine*, 43l, (1970), 103.
[46] Proposal 763, *Mathematics Magazine*, 43, (1970), 166.
[47] Proposal 775, *Mathematics Magazine*, 43, (1970), 278.
[48] Proposal 806, *Mathematics Magazine*, 44, (1971), 228.
[49] Proposal 839, *Mathematics Magazine*, 45, (1972), 228.
[50] Proposal 859, *Mathematics Magazine*, 46, (1973), 103.
[51] Proposal 916, *Mathematics Magazine*, 47, (1974), 286.
[52] Proposal 963, *Mathematics Magazine*, 49, (1976), 43.
[53] Proposal 998, *Mathematics Magazine*, 49, (1976), 252.
[54] Proposal 1197, *Mathematics Magazine*, 57, (1984), 238.
[55] Proposal 1206, *Mathematics Magazine*, 58, (1985), 46.
[56] Proposal 1211, *Mathematics Magazine*, 58, (1985), 111.
[57] Proposal 1298, *Mathematics Magazine*, 61, (1988), 195.
[58] Proposal 1305, *Mathematics Magazine*, 61, (1988), 261.
[59] Proposal 1327, *Mathematics Magazine*, 62, (1989), 274.
[60] Proposal 1356, *Mathematics Magazine*, 63, (1990), 274.
[61] Proposal 1371, *Mathematics Magazine*, 64, (1991), 132.
[62] Proposal 1377, *Mathematics Magazine*, 64, (1991), 197.
[63] Proposal 1405, *Mathematics Magazine*, 65, (1992), 265.

Proposal 208, *Mathematics Magazine*, 28, (1954-1955), 27.

208. *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Evaluate the following trigonometric expressions without using numerical tables:

$$A = \cos 5^\circ \cos 10^\circ \cos 15^\circ \cdots \cos 75^\circ \cos 80^\circ \cos 85^\circ,$$

$$B = \cos 1^\circ \cos 3^\circ \cos 5^\circ \cdots \cos 85^\circ \cos 87^\circ \cos 89^\circ,$$

$$C = \cos 4^\circ \cos 8^\circ \cos 12^\circ \cdots \cos 80^\circ \cos 84^\circ \cos 88^\circ.$$

Proposal 217, *Mathematics Magazine*, 28, (1954-1955), 103.

217. *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Prove that a necessary and sufficient condition for the convex polygon $A_1 A_2 A_3 A_4$ to be inscriptable is that:

$$D = \begin{vmatrix} A_1 A_1 & A_1 A_2 & A_1 A_3 & A_1 A_4 \\ A_2 A_1 & A_2 A_2 & A_2 A_3 & A_2 A_4 \\ A_3 A_1 & A_3 A_2 & A_3 A_3 & A_3 A_4 \\ A_4 A_1 & A_4 A_2 & A_4 A_3 & A_4 A_4 \end{vmatrix} = 0$$

where A_{ij} denotes the distance between the vertices A_i and A_j if $j > i$, and $A_j A_i = -A_i A_j$.

Proposal 227, *Mathematics Magazine*, 28, (1954-1955), 160.

227. *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Let $A_1 B_1$, $A_2 B_2$ and $A_3 B_3$ be three bars of lengths l_1, l_2 and l_3 with weights W_1, W_2 and W_3 respectively. The ends B_1, B_2 and B_3 rest on a horizontal surface while the other ends A_1, A_2 and A_3 are supported by the bars $A_3 B_3$, $A_1 B_1$ and $A_2 B_2$ respectively. Find the reactions R_1, R_2 and R_3 at B_1, B_2 and B_3.

Proposal 234, *Mathematics Magazine*, 28, (1954-1955), 234.

234. *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Given an m by n rectangular lattice containing mn points, find the total number of (a) squares, (b) rectangles having vertices at the points of the lattice. Consider $m \geq n$.

Proposal 242, *Mathematics Magazine*, 28, (1954-1955), 284.

242. *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Let A', B', C' be the points dividing the sides of triangle ABC in the ratio k, and let A'', B'', C'' be the points dividing the sides of triangle $A'B'C'$ in the ratio $1/k$. Prove that the triangle $A''B''C''$ is homothetic with the original triangle ABC.

Proposal 248, *Mathematics Magazine*, 29, (1955-1956), 46.

248. *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Let Γ_1 and Γ_2 be two plane curves. Let t be a variable line intersecting these curves at the points M_1, M_2 where the tangents t_1 and t_2 to the curves are parallel to each other. Prove that the centers of curvature C_1 and C_2 of Γ_1 and Γ_2 at M_1 and M_2 are collinear with the characteristic point C of the straight line t.

Proposal 258, *Mathematics Magazine*, 29, (1955-1956), 163.

258. *Proposed by Huseyin Demir, Zonguldak, Turkey.*

A triangle ABC inscribed in a circle varies such that AB and AC keep fixed directions. Find the locus of the orthocenter H.

Proposal 266, *Mathematics Magazine*, 29, (1955-1956), 222.

266. *Proposed by Huseyin Demir, Zonguldak, Turkey.*

If M and M' are points inverse to each other with respect to the circumcircle of a triangle ABC, then prove that:

$$\angle BMC + \angle BM'C = 2\angle A$$

$$\angle CMA + \angle CM'A = 2\angle B$$

$$\angle AMB + \angle AM'B = 2\angle C$$

Proposal 298, *Mathematics Magazine*, 30, (1956-1957), 164.

298. *Proposed by Huseyin Demir, Kandilli, Bolgesi, Turkey.*

Let $y = f(x)$ be a curve with the following properties

a) $f(x) = f(-x)$

b) $f'(x) > 0$ for $x > 0$

c) $f''(x) > 0$

Determine the weight per unit length $w(x)$ at the point (x, y) such that when the curve is suspended under gravity by any two points on it, the curve will keep its original shape.

Proposal 304, *Mathematics Magazine*, 30, (1956-1957), 223.

304. *Proposed by Huseyin Demir, Kandilli Bolgesi, Turkey.*

Let ABC be a triangle, $AB \neq AC$, inscribed in a circle O, and let K be the point where the exterior angle bisector of A meets O. A variable circle with center at K meets AB, AC at E and F respectively, such that A is not an interior point of KEF. Find the limiting position m of the common point M of EF, BC as EF approaches BC.

Proposal 334, *Mathematics Magazine*, 31, (1957-1958), 228.

334. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Find the simplest expression for the area S enclosed by the arc AM of a cycloid, the arc TM of the rolling circle Ω (a) and the base line segment AT.

Proposal 349, *Mathematics Magazine*, 32, (1958-1959), 47.

349. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

If $ABCD$, $AEBK$ and $CEFG$ are squares of the same orientations, prove that B bisects DF.

Proposal 372, *Mathematics Magazine*, 32, (1958-1959), 220.

372. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Prove the identity

$$\sin^2(\theta_1+\theta_2+\cdots+\theta_n) = \sin^2\theta_1+\cdots+\sin^2\theta_n + 2\sum_{1\leq i<j\leq n}^{n} \sin\theta_i \sin\theta_j \cos(\theta_1+2\theta_{i+1}+\cdots+2\theta_{j-1}+\theta_j).$$

Proposal 380, *Mathematics Magazine*, 32, (1958-1959), 278.

380. ***Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.***

Solve the system of equations

$$x(z-a)+u(x+u) = 0$$

$$y(x-b)+u(y+u) = 0$$

$$z(y-c)+u(z+u) = 0$$

where $abc \neq 0$ and $a^{-1}+b^{-1}+c^{-1} = u^{-1}$.

Proposal 384, *Mathematics Magazine*, 33, (1969-1960), 51.

384. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Let (a_{ij}) be a matrix of nth order the sum of the elements of whose rows equals 1. Prove that the totality $[(a_{ij})]$ form a group of infinite order.

Proposal 398, *Mathematics Magazine*, 33, (1969-1960), 165.

398. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Determine the roots of the equations $x^2+y_1x+y_2 = 0$, $y^2+x_1y+x_2 = 0$ where the coefficients (real numbers) in one equation are the roots of the other.

Proposal 407, *Mathematics Magazine*, 33, (1959-1960), 225.

407. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

The twelve edges of a cube are made of wires of one ohm resistance each. The cube is inserted into an electrical circuit by:

a) two adjacent vertices,

b) two opposite vertices of a face,

c) two opposite vertices of the cube.

Which offers the least resistance?

Proposal 415, *Mathematics Magazine*, 33, (1959-1960), 296.

415. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Prove

$$\sum_{p=0}^{n} \binom{n}{p} \cos(p)x \sin(n-p)x = 2^{n-1} \sin nx \ .$$

Proposal 419, *Mathematics Magazine*, 34, (1960-1961), 49.

419. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Determine the path in a vertical plane such that when a particle moved, under gravity, with an initial velocity v_0 from a point of the path, the particle maintained a constant speed along the path. Assume no friction.

Proposal 425, *Mathematics Magazine*, 34, (1960-1961), 109.

425. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

If $n-1$ and $n+1$ are twin prime numbers, prove that $3\phi(n) \leqq n$ where ϕ denotes Euler's ϕ-function.

Proposal 437, *Mathematics Magazine*, 34, (1961), 174.

437. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Prove or disprove the statement: The number of odd coefficients in the binomial expansion of $(a+b)^{[n]}$ is a power of 2, the exponent $[n]$ being the number of 1's appearing in the expression of n in the binary number system.

Proposal 440, *Mathematics Magazine*, 34, (1961), 237.

440. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Consider a packing of circles of radius r such that each is tangent to its six surrounding circles. Let a larger circle of radius R be drawn concentric with one of the small circles. If N is the number of small circles contained in the larger circle, prove that

$$N = 1 + 6n + 6 \sum_{p=1}^{n} [\tfrac{1}{2}(\sqrt{4n^2 - 3p^2} - p)]$$

where $n = [\frac{1}{2}(\frac{R}{r} - 1)]$, the square brackets designating the greatest integer function.

Proposal 458, *Mathematics Magazine*, 34, (1961), 364.

458. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

A student used DeMoivre's theorem incorrectly as

$$(\sin \alpha + i \cos \alpha)^n = \sin n\alpha + i \cos n\alpha \,.$$

For what values of α does the equation hold for every integer n?

Proposal 472, *Mathematics Magazine*, 35, (1962), 55.

472. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Let (C) be a conic and M be a variable point on it. Let T be the point symmetric to M with respect to the main axis, and t the tangent line at T. Denote the intersection of the perpendicular from M to t with the line joining T to the center of the conic by I. If M' is symmetric to M with respect to I, prove that

1. The locus of M' is another conic (C') of the same kind as (C).
2. The conics (C) and (C') are confocal.

Proposal 487, *Mathematics Magazine*, 35, (1962), 186.

487. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Find the square root of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Proposal 498, ***Mathematics Magazine***, 35, (1962), 309.

498. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

If m and n are integers and δ, D are their g.c.d. and l.c.m. respectively, and $d(n)$ denotes the number of divisors of n, $\phi(n)$ being the Euler function, prove that:

$$d(m)d(n) = d(\delta)d(D) \tag{1}$$

$$\phi(m)\phi(n) = \phi(\delta)\phi(D) \tag{2}$$

Proposal 509, ***Mathematics Magazine***, 36, (1963), 133.

509. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the cryptarithm

$$\begin{array}{r} U\ N\ I\ T\ E\ D \\ S\ T\ A\ T\ E\ S \\ \hline A\ M\ E\ R\ I\ C\ A \end{array}$$

in the base 11, introducing the digit α.

Proposal 517, ***Mathematics Magazine***, 36, (1963), 197.

517. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let F and d be the focus and directrix of a parabola. If M and N are any two points on the parabola and M', N' are their respective projections on d, show that

$$\frac{\text{Area } FMN}{\text{Area } N'M'\text{MN}} = \text{Constant.}$$

Proposal 529, *Mathematics Magazine*, 36, (1963), 264.

529. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

A cycloid (cardioid) rolls on a straight line without sliding. Prove that the locus of the center of curvature of the curve at the point of tangency is a circle (ellipse).

Proposal 537, *Mathematics Magazine*, 37, (1964), 55.

537. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Determine the relative positions of an equilateral triangle and a square inscribed in the same circle so that their common area shall be an extremum.

Proposal 544, *Mathematics Magazine*, 37, (1964), 119.

544. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the cryptarithm (alphametic)

$$ONE+TWO+SIX=NINE$$

in the base 10, with the following conditions:

a) $ONE<TWO<SIX$

b) $2|TWO$, $6|SIX$, $9|NINE$ where $a|b$ means "a divides b."

Proposal 563, *Mathematics Magazine*, 37, (1964), 276.

563. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let A, B', A', B be four consecutive vertices of a regular hexagon. If M is an arbitrary point of the circumcircle (in particular on arc $A'B'$) and MA, MB intersect BB' and AA' in the points E and F respectively, then prove that:

$$\text{(a)} \quad \measuredangle MEF = 3\measuredangle MAF$$

$$\text{(b)} \quad \measuredangle MFE = 3\measuredangle MBE.$$

Proposal 572, *Mathematics Magazine*, 38, (1965), 52.

572. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

To the memory of President Kennedy. Mr. J. F. Kennedy was killed on November 22, 1963. That is, on the day 11-22-1963. Solve the cryptarithm

$$JF\cdot(KEN + NEDY) = (11 + 22)\cdot 1963$$

in the decimal system.

Proposal 587, *Mathematics Magazine*, 38, (1965), 179.

587. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Prove the following inequality

$$\left(\frac{\theta + \sin\theta}{\pi}\right)^2 + \cos^4\frac{1}{2}\theta < 1, \qquad (-\pi < \theta < +\pi).$$

Proposal 599, *Mathematics Magazine*, 38, (1965), 241.

599. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

If a, b, and c are any three vectors in 3-space, then show that the vectors

$$ax(bxc),\ bx(cxa),\ cx(axb)$$

are linearly dependent.

Proposal 600, *Mathematics Magazine*, 38, (1965), 317.

600. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

If the area of a triangle ABC is S and the areas of the in- and ex-contact triangles are T, T_a, T_b, T_c, then show that

$$T_a + T_b + T_c - T = 2S \tag{1}$$

$$T_a^{-1} + T_b^{-1} + T_c^{-1} - T^{-1} = 0. \tag{2}$$

Proposal 609, *Mathematics Magazine*, 39, (1966), 69.

609. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the following cryptarithm in the decimal system:

$$4 \cdot NINE = 9 \cdot FOUR$$

Proposal 628, *Mathematics Magazine*, 39, (1966), 246.

628. *Proposed by B. Suer and Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the alphametic,

$$\mathrm{COS}^2 + \mathrm{SIN}^2 = \mathrm{UNO}^2$$

in the decimal system.

Proposal 639, *Mathematics Magazine*, 39, (1966), 306.

639. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let $ABCD$ be a convex quadrangle and P be the intersection of diagonals AC and BD. Let the distance of P from the sides $AB=a$, $BC=b$, $CD=c$, $DA=d$ be x, y, z, and t respectively. Prove that

$$x + y + z + t < \tfrac{3}{4}(a + b + c + d).$$

Proposal 649, *Mathematics Magazine*, 40, (1967), 100.

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649. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the cryptarithm

$$\begin{array}{r} THREE \\ +\quad FOUR \\ \hline SEVEN \end{array}$$

in the decimal system such that:

3 does not divide $T\,H\,R\,E\,E$ in which the digit 3 is missing;
4 does not divide $F\,O\,U\,R$ in which the digit 4 is missing;
7 does not divide $S\,E\,V\,E\,N$ in which the digit 7 is missing.

Proposal 680, ***Mathematics Magazine***, 41, (1968), 42.

680. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let E be an ellipse and t', t'' be two variable parallel tangents to it. Consider a circle C, tangent to t', t'' and to E externally. Show that the locus of the center of C is a circle.

Proposal 724, ***Mathematics Magazine***, 42, (1969), 96.

724. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Find the probability that for a point P taken at random in the interior of a triangle ABC $(a \geqq b \geqq c)$, the distances of P from the sides of ABC form the lengths of sides of a triangle.

Proposal 738, ***Mathematics Magazine***, 42, (1969), 214.

738. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

There is a river with parallel and straight shores. A is located on one shore and B on the other, with $AB = 72$ miles. A ferry boat travels the straight path AB from A to B in four hours and from B to A in nine hours. If the boat's speed on still water is $v = 13$ mph, what is the velocity of the flow?

Proposal 743, *Mathematics Magazine*, 42, (1969), 267.

743. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let P be an interior point of a regular tetrahedron, $T \equiv A_1A_2A_3A_4$, with $p_i = PA_i$, and let x_{ij} denote the distance of P from the edge A_iA_j. Then prove

$$\sum_{i=1}^{4} p_i \geqq 2\sqrt{3}/3 \sum_{i<j} x_{ij},$$

equality holding if and only if P is at the center O of T.

Proposal 756, *Mathematics Magazine*, 43, (1970), 103.

756. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Determine closed and centrally symmetric curves C, other than circles, such that the product of two perpendicular radius vectors (issued from the center) be a constant.

Proposal 763, *Mathematics Magazine*, 43, (1970), 166.

763. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Prove:

$$\left(1+\frac{1}{3^{10}}+\frac{1}{5^{10}}+\cdots\right)=\left(1+\frac{1}{3^4}+\frac{1}{5^4}\cdots\right)\left(1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\cdots\right).$$

Proposal 775, *Mathematics Magazine*, 43, (1970), 278.

775. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Prove $\int_0^1 \sqrt[q]{1-x^p}\,dx = \int_0^1 \sqrt[p]{1-x^q}\,dx$, where $p, q > 0$.

Proposal 806, ***Mathematics Magazine***, 44, (1971), 228.

806. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let H be the orthocenter of an isosceles triangle ABC, and let AH, BH, and CH intersect the opposite sides in D, E, and F, respectively. Prove that the incenters of the right triangles HBD, HDC, HCE, HEA, HAF, and HFB lie on a conic.

Proposal 839, ***Mathematics Magazine***, 45, (1972), 228.

839. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Given three boxes each containing w white balls and r red balls identical in shape. Take a ball from the first box and put it in the second box, then take a ball from the second box and put it in the third, and finally take a ball from the third box and put it in the first. Find the probability that the boxes have their original contents as to color.

Proposal 859, ***Mathematics Magazine***, 46, (1973), 103.

859. *Proposed by B. Suer and H. Demir, Middle East Technical University, Ankara, Turkey.*

Solve the cryptarithm

$$THREE + NINE = EIGHT + FOUR.$$

Proposal 916, ***Mathematics Magazine***, 47, (1974), 286.

916. *Proposed by H. Demir, M.E.T.U., Ankara, Turkey.*

Let XYZ be the pedal triangle of a point P with regard to the triangle ABC. Then find the trilinear coordinates x, y, z of P such that

$$YA + AZ = ZB + BX = XC + CY.$$

Proposal 963, *Mathematics Magazine*, 49, (1976), 43.

963. Characterize convex quadrilaterals with sides $a, b, c,$ and d such that

$$\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix} = 0$$

[*Hüseyin Demir, Ankara, Turkey.*]

Proposal 998, *Mathematics Magazine*, 49, (1976), 252.

998. Characterize all triangles in which the triangle whose vertices are the feet of the internal angle bisectors is a right triangle. [*Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*]

Proposal 1197, *Mathematics Magazine*, 57, (1984), 238.

1197. Characterize the triangles of which the midpoints of the altitudes are collinear. [*Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*]

Proposal 1206, *Mathematics Magazine*, 58, (1985), 46.

1206. Let ABC be a triangle with sides a, b, and c and semiperimeter s. Let the side BC be subdivided using the points $B = P_0, P_1, \dots, P_{n-1}, P_n = C$ in order. If r_i is the inradius of triangle $AP_{i-1}P_i$ for $i = 1, \dots, n$, prove that

$$r_1 + \cdots + r_n < \frac{1}{2} h_a \ln \frac{s}{s-a},$$

where h_a is the length of the altitude from vertex A. [*Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*]

Proposal 1211, *Mathematics Magazine*, 58, (1985), 111.

1211. Find the locus of points under which an ellipse is seen under a constant angle. [*Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*]

Proposal 1298, *Mathematics Magazine*, 61, (1988), 195.

1298. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

A quadrilateral $ABCD$ is circumscribed about a circle, and P, Q, R, S are the points of tangency of sides AB, BC, CD, DA respectively. Let $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DA|$, and $p = |QS|$, $q = |PR|$. Show that

$$\frac{ac}{p^2} = \frac{bd}{q^2}.$$

Proposal 1305, *Mathematics Magazine*, 61, (1988), 261.

1305. *Proposed by H. Demir and C. Tezer, Middle East Technical University, Ankara, Turkey.*

Let $P_0 = B, P_1, P_2, \ldots, P_n = C$ be points, taken in that order, on the side BC of the triangle ABC. If r, r_i and h denote respectively the inradii of the triangles ABC, $AP_{i-1}P_i$ and the common altitude, prove that

$$\prod_{i=1}^{n} \left(1 - \frac{2r_i}{h}\right) = 1 - \frac{2r}{h}.$$

Proposal 1327, *Mathematics Magazine*, 62, (1989), 274.

1327. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let the sides PQ, QR, RS, SP of a convex quadrangle $PQRS$ touch an inscribed circle at A, B, C, D and let the midpoints of the sides AB, BC, CD, DA be E, F, G, H. Show that the angle between the diagonals PR, QS is equal to the angle between the bimedians EG, FH.

Proposal 1356, *Mathematics Magazine*, 63, (1990), 274.

1356. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let P, Q be points taken on the side BC of a triangle ABC, in the order B, P, Q, C. Let the circumcircles of PAB, QAC intersect at M $(\neq A)$ and those of PAC, QAB at N. Show that A, M, N are collinear if and only if P and Q are symmetric in the midpoint A' of BC.

Proposal 1371, *Mathematics Magazine*, 64, (1991), 132.

1371. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let A, B, and C be vertices of a triangle and let D, E, and F be points on the sides BC, AC, and AB, respectively. Let U, X, V, Y, W, Z be the midpoints of, respectively, BD, DC, CE, EA, AF, FB. Prove that

$$\text{Area}(\triangle UVW) + \text{Area}(\triangle XYZ) - \frac{1}{2}\text{Area}(\triangle DEF)$$

is a constant independent of D, E, and F.

Proposal 1377, *Mathematics Magazine*, 64, (1991), 197.

1377. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let DEF be a variable triangle inscribed in triangle ABC, and let U, X, V, Y, W, Z be the midpoints of the line segments BD, DC, CE, EA, AF, and FB, respectively.

Show that the expression

$$|UVW| + |XYZ| - \tfrac{1}{2}|DEF|$$

for areas is constant.

Proposal 1405, *Mathematics Magazine*, 65, (1992), 265.

1405. *Proposed by Hüseyin Demin, Middle East Technical University, Ankara, Turkey.*

Two circles inscribed in distinct angles of a triangle are *isogonally related* if the tangents from the third vertex not coinciding with the sides are symmetric with respect to the bisector of the third angle. Given three circles inscribed in distinct angles of a triangle, prove that if any two of the three pairs of circles are isogonally related then so is the third pair.

7 Solutions of Proposals

Solution to Proposal 208:
Mathematics Magazine, 28, (1954-1955), 27.

I. *Solution by E. P. Starke, Rutgers University.*

$$\sqrt{2}\, A = \cos 5^\circ \cos 10^\circ \cdots \cos 40^\circ \sin 40^\circ \cdots \sin 10^\circ \sin 5^\circ$$
$$= \sin 10^\circ \sin 20^\circ \cdots \sin 80^\circ / 2^8,$$

by use of the double-angle formula. A repetition of the same device gives: $2^{12}\sqrt{2}\, A = \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \sqrt{3}\, k/2$, say, where $k = \sin 20^\circ(\sin 40^\circ \sin 80^\circ) = \sin 20^\circ(\sin^2 60^\circ - \sin^2 20^\circ)$ $= \frac{1}{4}(3 \sin 20^\circ - 4 \sin^3 20^\circ) = \frac{1}{4} \sin 60^\circ = \sqrt{3}/8$. So $A = 3 \cdot 2^{-33/2}$.

Similarly $B = \cos 1^\circ \sin 1^\circ \cos 3^\circ \sin 3^\circ \cdots$ $\cos 43^\circ \sin 43^\circ \cos 45^\circ$. $2^{22}\sqrt{2}\, B = \sin 2^\circ \sin 6^\circ \sin 10^\circ \ldots \sin 86^\circ = C$.

Let $x = \cos 2^\circ \cos 6^\circ \cos 10^\circ \ldots \cos 86^\circ$.

Then $2^{22} \cdot 2^{22}\sqrt{2}\, B \cdot x = \sin 4^\circ \sin 12^\circ \sin 20^\circ \cdots \sin 172^\circ$
$$= \sin 4^\circ \sin 8^\circ \sin 12^\circ \sin 16^\circ \cdots \sin 88^\circ = x,$$

whence $B = 2^{-89/2}$ and $C = 2^{-22}$.

II. *Solution by H. M. Feldman, Washington University, St. Louis Missouri.*

From the identities

$$x^{2n} - 1 = (x^2 - 1)(x^{2n} + x^{2n-1} + \cdots + 1) = (x^2 - 1)\prod_{k=1}^{n-1}(x^2 - 2x\cos\frac{k\pi}{n} + 1) \quad \text{and}$$

$$x^{2n+1} = (x+1)(x^{2n} - x^{2n-1} + \cdots + 1) = (x+1)\prod_{k=1}^{n}(x^2 - 2x\cos\frac{2k-1}{2n+1} + 1)$$

we get, by letting $x = \pm 1$, the following relations:

$$\prod_1^{n-1}\sin\frac{k\pi}{2n} = \prod_1^{n-1}\cos\frac{k\pi}{n} = 2^{-n+1}\sqrt{n}\ ;$$

$$\prod_1^{n-1}\sin\frac{k\pi}{n}\ 2^{-n+1}n;\ \prod_1^{n}\sin\frac{2k-1}{2(2n+1)}\pi = 2^{-n}$$

$$\text{and} \prod_1^{n}\cos\frac{2k-1}{2(2n+1)} = \prod_1^{n}\sin\frac{2k-1}{2n+1}\pi = 2^{-n}\sqrt{2n+1}\,.$$

By means of these relations, we find:

$$A = 2^{-17}(3\sqrt{2})$$
$$B = 2^{-45}\sqrt{2}$$
$$C = 2^{-22}$$

Also solved by Leon Bankoff, Los Angeles, California; Kwan Moon (partially), Mississippi State College; George Mott, Republic Aviation Corp., New York; T. F. Mulcrone, St. Charles College, Louisiana; L. A. Ringenberg, Eastern Illinois State College; Chih-yi Wang, University of Minnesota; Hazel S. Wilson, Jacksonville State College, Alabama and the proposer.

Solution to Proposal 217:
Mathematics Magazine, 28, (1954-1955), 103.

Solution by H. M. Feldman, St. Louis, Missouri

Since A_iA_i must clearly be zero, the determinant is skew-symmetric and its value is

$$[(A_1A_2)(A_3A_4) + (A_1A_4)(A_2A_3) + (A_1A_3)(A_2A_4)]^2$$

The vanishing of the expression within the brackets is a necessary and sufficient condition for the quadrilateral to be inscriptable in a circle (Ptolemy's Theorem).

Also solved by Ben K. Gold, Los Angeles City College; M. S. Klamklin, Polytechnic Institute of Brooklyn; E. P. Starke; Rutgers University; Chih-yi Wang, University of Minnesota and the proposer.

Solution to Proposal 227:
***Mathematics Magazine*, 28, (1954-1955), 160.**

Forces In Equilibrium

227. [January 1955] *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Let A_1B_1, A_2B_2 and A_3B_3 be three bars of lengths l_1, l_2 and l_3 with weights W_1, W_2 and W_3 respectively. The ends B_1, B_2 and B_3 rest on a horizontal surface while the other ends A_1, A_2 and A_3 are supported by the bars A_3B_3, A_1B_1 and A_2B_2 respectively. Find the reactions R_1, R_2 and R_3 at B_1, B_2 and B_3.

Solution by the proposer. Let the reactions of the bars at the ends A_1, A_2, A_3 be denoted by r_1, r_2, r_3 and the lengths A_1A_2, A_2B_1: A_2A_3, A_3B_2; A_3A_1, A_1B_3 by a_1, b_1; a_2, b_2; a_3, b_3 respectively.

Then considering the equilibrium of one of the bars, say A_1B_1, we have by taking moments of the forces r_1, r_2, W_1, R_1 at the point B_1:

$$\tfrac{1}{2}\, l_1 W_1 - l_1 r_1 + b_1 r_2 = 0.$$

Setting $b_i = k_i l_i$ $(i = 1,2,3)$ and considering the other equations corresponding to the two other bars, we get the system of equations with unknowns r_1, r_2, r_3:

$$r_1 - k_1 r_2 = \tfrac{1}{2} W_1$$
$$r_2 - k_2 r_3 = \tfrac{1}{2} W_2$$
$$r_3 - k_3 r_1 = \tfrac{1}{2} W_3$$

The determinant of this system being

$$D = \begin{vmatrix} 1 & -k_1 & 0 \\ 0 & 1 & -k_2 \\ -k_3 & 0 & 1 \end{vmatrix} = 1 - k_1 k_2 k_3$$

we have

$$r_1 = (W_1 + k_1 W_2 + k_1 k_2 W_3)/2D$$
$$r_2 = (W_2 + k_2 W_3 + k_2 k_3 W_1)/2D$$
$$r_3 = (W_3 ++ k_3 W_1 + k_3 k_1 W_2)/2D$$

and

$$R_1 = W_1 - r_1 + r_2 = W_1 - (W_1 + k_1 W_2 + k_1 k_2 W_3)/2D + W_2 + k_2 W_3 + k_2 k_3 W_1)/2D$$

$$R_1 = [(1 + k_2 k_3 - 2k_1 k_2 k_3)W_1 + (k_1 - 1)W_2 + k_2 k_1 - 1)W_3]/2(1 - k_1 k_2 k_3)$$
$$R_2 = [k_3(k_2 - 1)W_1 + (1 + k_3 k_1 - 2k_1 k_2 k_3)W_2 + (k_2 - 1)W_3]/2(1 - k_1 k_2 k_3)$$
$$R_3 = [(k_3 - 1)W_1 + k_1(k_3 - 1)W_2 + (1 + k_1 k_2 - 2k_1 k_2 k_3)W_3]/2(1 - k_1 k_2 k_3)$$

Also solved by George R. Mott, Republic Aviation Company.

Solution to Proposal 234:
Mathematics Magazine, 28, (1954-1955), 234.

Solution by the proposer. We distinguish two kinds of squares. A square is an N- or L-square according as their sides are or are not parallel to the sides of the lattice.

Every L-square is inscribed in a unique N-square. By a $p \times p$ N-square we mean one having p points on each of its sides. In such a square are inscribed evidently $p-2$ L-squares. Including the N-square itself the number is $p-1$.

The number of $p \times p$ N-squares is easily seen to be $(m - p + 1)(n - p + 1)$. Hence the number of $p \times p$ N-squares together with L-squares inscribed in them is $(p - 1)(m - p + 1)(n - p + 1)$. Hence the required total number of squares is given by

$$N = \sum_{p=2}^{m} (p - 1)(m - p + 1)(n - p + 1)$$

$$= mn \sum (p - 1) - (m + n) \sum (p - 1)^2 + \sum (p - 1)^3$$

$$= mn \frac{n(n - 1)}{2} - (m + n)\frac{n(n - 1)(2n - 1)}{6} + \frac{n^2(n - 1)^2}{4}$$

$$= \frac{n(n - 1)}{12}\left[6mn - 2(m + n)(2n - 1) + 3n(n - 1)\right]$$

$$= n(n^2 - 1)(2m - n)/12.$$

No solution of the rectangular case has been received. Solutions restricting the squares and rectangles to those with sides parallel to the lines of lattice points were received from *Julian H. Braun, White Sands Proving Ground and E. P. Starke, Rutgers University.* Braun noted that the restricted case was a variation of Problem E 1127 of the *American Mathematical Monthly.*

Solution to Proposal 242:
Mathematics Magazine, 28, (1954-1955), 284.

Solution by P. W. Allen Raine, Newport News High School, Newport News, Virginia. Let A, B, C, A', B', C', A'', B'', C'' represent the vector coordinates of the respective points and k, a scalar quantity. Thus

$$A' = \frac{kB + C}{k + 1}, \quad B' = \frac{kC + A}{k + 1}, \quad C' = \frac{kA + B}{k + 1}$$

and

$$A'' = \frac{B' + kC'}{1 + k} = \frac{kC + A + k^2A + kB}{(1 + k)^2},$$

$$B'' = \frac{B' + kA'}{1 + k} = \frac{kA + B + k^2B + kC}{(1 + k)^2},$$

$$C'' = \frac{A' + kB'}{1 + k} = \frac{kB + C + k^2C + kA}{(1 + k)^2}$$

Now we can easily show that

$$A'' - B'' = \frac{1 - k + k^2}{(1 + k)^2}(A - B),$$

$$B'' - C'' = \frac{1 - k + k^2}{(1 + k)^2}(B - C),$$

$$C'' - A'' = \frac{1 - k + k^2}{(1 + k)^2}(C - A)$$

which tells us that the sides of the two triangles are parallel and hence the triangles are homothetic, the homothetic ratio being

$$\frac{1 - k + k^2}{(1 + k)^2}$$

Also solved by Maimouna Edy, Hull, P. Q., Canada; M. S. Klamkin, Polytechnic Institute of Brooklyn; Chih-yi Wang, University of Minnesota and the proposer.

Solution to Proposal 248:
Mathematics Magazine, 29, (1955-1956), 46.

Solution by the proposer. Considering the new position t' of t very close to t, we have $M_1M_1'/\sin\Delta\theta = C'M_1'/\sin M_1$, where M_1' is close to M_1 on Γ_1, and $\Delta\theta = (t,t')$; the angle between t and t'.

Infinitesimally

$$ds_1/d\theta = CM_1/\sin\mu_1, \qquad \mu_1 = (t,t_1) \pm \pi$$

and similarly

$$ds_2/d\theta = CM_2/\sin\mu_2, \qquad \mu_2 = (t,t_2) \pm \pi$$

Having $\sin\mu_1 = \sin\mu_2$, as t_1 is parallel to t_2, we get

$$ds_1/CM_1 = ds_2/CM_2$$

which in turn yields

$$(ds_1/d\alpha)CM_1 = (ds_2/d\alpha)CM_2$$

i.e.

$$R_1/CM_1 = R_2/CM_2$$

where $d\alpha$ is the infinitesimal angle relative to the parallel normals at M_1, M_2, and R_1, R_2 the corresponding radii of curvature. The last equality proves the statement.

Also solved by Richard K. Guy, University of Malaya, Singapore and Chih-yi Wang, University of Minnesota.

Solution to Proposal 258:
Mathematics Magazine, 30, (1956-1957), 47.

An Orthocentric Locus

258. [January 1956] *Proposed by Huseyin Demir, Zonguldak, Turkey.*

A triangle ABC inscribed in a circle varies such that AB and AC keep fixed directions. Find the locus of the orthocenter H.

1. *Solution by Major H. S. Subba Rao, Defense Science Organization, New Delhi, India.* The vertical angle A and the base BC are fixed in magnitude. Let $A_1B_1C_1$ be the isosceles triangle satisfying the conditions imposed on ABC. Let P be the mid-point of the smaller of the two arcs AC of the circum-circle and similarly Q the mid-point of the arc AB. Let O be the centre of the circle. The points P and Q are fixed.

Take the diameter through A, as the y-axis and the perpendicular diameter as the x-axis. With reference to these axes we can represent any point on the circle ABC by the parametric representation $a \cos t$, $a \sin t$.

Let $B_1 \equiv (t_2)$, $C_1 \equiv (t_3)$, $P \equiv (t_4)$ and $Q \equiv (t_5)$.

Noting that angle $A_1B_1C_1$ angle $A_1C_1B_1 = 90^\circ - A/2$ it can be easily shown that $t_2 = \dfrac{3\pi}{2} - A$, $t_3 = \dfrac{3\pi}{2} + A$, $t_4 = \dfrac{A}{2}$, $t_5 = 2\pi - \dfrac{A}{2}$.

In any position of the triangle ABC let $t = \measuredangle B_1OB = \measuredangle C_1OC$. Then $B \equiv (t_2 + t)$ and $C \equiv (t_3 + t)$. Further, BH being perpendicular to AC is parallel to OP and similarly CH is parallel to OQ.
The equations to BH and CH are easily found to be

$$x \operatorname{Sin} \frac{A}{2} - y \cos \frac{A}{2} = a \cos \left(t - \frac{3A}{2}\right)$$

and

$$x \operatorname{Sin} \frac{A}{2} + y \cos \frac{A}{2} = a \cos \left(t + \frac{3A}{2}\right) .$$

Eliminating t between the two equations, the locus of H is found to be

$$\frac{x^2 \operatorname{Sin}^2 \frac{A}{2}}{a^2 \cos^2 \frac{3A}{2}} + \frac{y^2 \cos^2 \frac{A}{2}}{a^2 \operatorname{Sin}^2 \frac{3A}{2}} = 1.$$

This is an ellipse with its centre at O and semi axes

$$\frac{a \operatorname{Cos} \frac{3A}{2}}{\operatorname{Sin} \frac{A}{2}} \quad \text{and} \quad \frac{a \operatorname{Sin} \frac{3A}{2}}{\operatorname{Cos} \frac{A}{2}}$$

(An interesting corollary to this is that the loci of the nine-point centre and centroid of the triangle ABC are also ellipses).

II. *Solution by the proposer*. Let OX, OY be the lines parallel to external and internal bisectors of A respectively. Let the altitude AH intersect these fixed lines at X, Y. Since AO, AH are equally inclined to the bisectors of A, we have $AX = AO = AY$. Hence $XY = 2R$ = const.

We may think then of XY as a rod of constant length having the ends moving on OX, OY. Now the angle A being constant, BC will envelop, or the mid-point D of BC will describe a circle with center O. Hence $AH = 2\ OD = 2\ R\ \cos A$ = const. This proves that H is a fixed point of the moving bar XAY. Hence H describes an ellipse.
The semi-diameters of the ellipse are easily determined:

$$a = HY + AY = HA = R(1 + 2\cos A), \qquad b = HX = XA - HA = R(1 - 2\cos A).$$

Also solved by J. W. Clawson, Collegeville, Pennsylvania; R. K. Guy, University of Malaya, Singapore; Sister M. Stephanie, Georgian Court College, New Jersey; Harry D. Ruderman, The Bronx, New York and Chih-yi Wang, University of Minnesota.

Solution to Proposal 266:
Mathematics Magazine, 30, (1956-1957), 105.

Points Inverse in a Circumcircle

266.[March 1956] *Proposed by Huseyin Demir, Zonguldak, Turkey.*

If M and M' are points inverse to each other with respect to the circumcircle of a triangle ABC then prove that:

$$\angle BMC + \angle BM'C = 2\angle A$$
$$\angle CMA + \angle CM'A = 2\angle B$$
$$\angle AMB + \angle AM'B = 2\angle C$$

I. *Solution by Richard K. Guy, University of Malaya, Singapore.* In triangles COM and $M'OC$ angle O is common and as $OM \cdot OM' = OC^2$ we have $\dfrac{OC}{OM} = \dfrac{OM'}{OC}$. Hence the triangles are similar and $\measuredangle OM'C = \measuredangle OCM$. In the same way $\measuredangle OM'B = \measuredangle OBM$. Adding these to $\measuredangle OMC$ and $\measuredangle OMB$ we have $\measuredangle BMC + \measuredangle BM'C = \pi - \measuredangle COM + \pi - \measuredangle BOM = \measuredangle BOC = 2\measuredangle A$.

Simmlarly we have $\measuredangle CMA + \measuredangle CM'A = 2\measuredangle B$ and $\measuredangle AMB = \measuredangle AM'B = 2\measuredangle C$.

II. *Solution by Maimouna Edy, Hull, PQ, Canada.* Represent points A, B, C, M, M' by complex numbers z_1, z_2, z_3, z, z' respectively. Let parentheses represent cross ratios and the bars the complex conjugate. We then have:

$$(1) \qquad (z_1, z_2, z_3, z) = \overline{(z_1, z_2, z_3, z')}$$

This says that the homographic transformation which sends z_1, z_2, z_3 into $1, 0, \infty$ respectively, that is the transformation of the given circle into the axis of reals, sends z and z' into two conjugate complex points.

Now equation (1) reads explicitly

$$\frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2} = \overline{\frac{z' - z_2}{z' - z_3}} \cdot \overline{\frac{z_1 - z_3}{z_1 - z_2}}$$

Evidently

$$\frac{z_2 - z}{z_3 - z} \div \overline{\frac{z_2 - z'}{z_3 - z'}} = \overline{\frac{z_3 - z_1}{z_2 - z_1}} \div \frac{z_3 - z_1}{z_2 - z_1}.$$

Therefore

$$\arg\left(\frac{z_2 - z}{z_3 - z}\right) + \arg\left(\frac{z_2 - z'}{z_3 - z'}\right) = -2\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = +2\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right)$$

This means that for oriented angles,

$$(\text{angle } \vec{MC}, \vec{MB}) + (\text{angle } \vec{M'C}, \vec{M'B}) = 2(\text{angle } \vec{AC}, \vec{AB}).$$

The oriented angles form an additive group isomorphic with the multiplicative group of the unit circle. In other words, we may take arbitrary measures of our angles and add them (mod 2π). The other two relations are proven similarly.

Bankoff's solution also noted the necessity for proper orientation of the angles.

Also solved by Leon Bankoff, Los Angeles, California; J.W. Clawson, Collegeville, Pennsylvania; and the proposer.

Solution to Proposal 298:
Mathematics Magazine, 31, (1957-1958), 56.

An Invariant Curve

298. [January 1957] *Proposed by Huseyin Demir, Kandilli, Bolgesi, Turkey.*

Let $y = f(x)$ be a curve with the following properties

a) $f(x) = f(-x)$

b) $f'(x) > 0$ for $x > 0$

c) $f''(x) = 0$

Determine the weight per unit length $w(x)$ at the point (x, y) such that when the curve is suspended under gravity by any two points on it, the curve will keep its original shape.

Solution by K.L. Cappel, Philadelphia, Pennsylvania. Assume the curve to be suspended at two arbitrary points A and B. Let the weight between A and the y intercept of the curve be W. Then at A, the tension in the curve can be resolved into vertical and horizontal components so that $W/H = \tan\theta$ or $W = H \ dy/dx$.

Now assume the right point of support to be moved from A to A'. If the curve is to retain its shape, there must be no change in the forces at A. This can only be the case if H is a constant. If ds is the length of the segment AA', and dW is its weight, then the weight per unit length will be

$$W_x = \frac{dW}{ds} = \frac{d\ W}{\sqrt{1 + (\frac{dy}{dx})^2}\ dx} \qquad or, \qquad W_x = H \cdot \frac{d^2y/dx^2}{\sqrt{1 + (\frac{dy}{dx})^2}}$$

which can be satisfied by any curve obeying the given conditions.

This problem is analagous to the problem of finding the optimum shape of a masonry arch, when the material of the arch is the only load to be supported, and it is desired to have the thrust load act along the neutral axis in order to eliminate bending moments.

Also solved by the proposer

Solution to Proposal 304:
***Mathematics Magazine*, 31, (1957-1958), 117.**

CIRCLES CONNECTED WITH TRIANGLE

304. [March 1957] *Proposed by Huseyin Demir, Kandill, Bolgesi, Turkey.*

Let ABC be a triangle, $AB \neq AC$, inscribed in a circle (O), and let K be the point where the exterior angle bisector of A meets (O). A variablè circle with center at K meets AB, AC at E and F respectively, such that A is not an interior point of KEF. Find the limiting position m of the common point M of EF, BC as EF approaches BC.

Solution by the Proposer. Let E', F' be the points where (K) meets AB, AC other than E, F. Let M' be the common point of BC with $E'F'$.

Applying the Menelaus theorem to ABC, considering EFM, $E'F'M'$ as transversals, we have

$$\frac{MB}{MC} \cdot \frac{FC}{FA} \cdot \frac{EA}{EB} = +1 \qquad\qquad \frac{M'B}{M'C} \cdot \frac{F'C}{F'A} \cdot \frac{E'A}{E'B} = +1$$

Multiplying these equalities member to member and observing that $EA = F'A$, $E'A = FA$ we get

$$\frac{MB}{MC} \cdot \frac{M'B}{M'C} = \frac{EB \cdot E'B}{FC \cdot F'C}$$

Since in the last ratio the numerator and denominator are the powers of B, C with respect to the circle (K), and since these powers are equal (K is equidistant from B and C) $MB:MC = M'C:M'B$ follows. Hence the points M and M' are symmetric points on BC. The limiting position m of M will also be symmetric point of m', the limiting position of M'. It is easy to see that m' is the foot of KA, the exterior angle bisector of A. Hence the construction of M follows immediately.

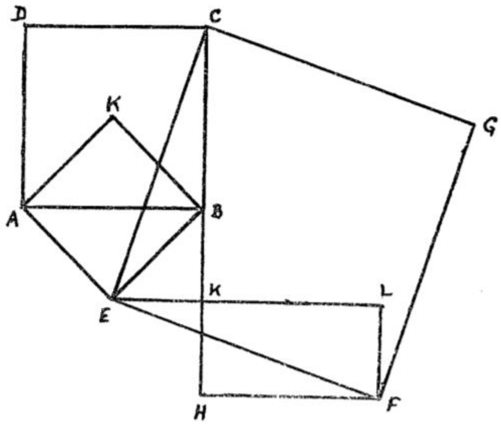
Solution to Proposal 349:
***Mathematics Magazine*, 32, (1958-1959), 223.**

A Bisector

349. [September 1958] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

If $ABCD$, $AEBK$ and $CEFG$ are squares of the same orientations, prove that B bisects DF.

Solution by Leon Bankoff, Los Angeles, California. Removing angle CEB from the right angles AEB and CEF, we find that angles BEF, AEC, DEB are equal. But $DE = EC = EF$. Hence the triangles BDE and FBE are congruent and $FB = BD$. The collinearity of F, B, D is established by the fact that angle $EBD = 90°$.



Also solved by Norman Anning, Alhambra, California; D.A.Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; J.W.Clawson, Collegeville, Pennsylvania; Norbert Jay, New York, New York; Joseph D.E.Konhauser, Haller, Raymond and Brown, Inc., State College, Pennsylvania; Arne Pleijel, Trollhattan, Sweden; William Sanders, Mississippi Southern College; C.W.Trigg, Los Angeles City College, Dale Woods, Idaho State College, and the proposer.

Solution to Proposal 334:
***Mathematics Magazine*, 32, (1958-1959), 106.**

An Irregular Area

334. [March 1958] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Find the simplest expression for the area S enclosed by the arc AM of a cycloid, the arc TM of the rolling circle $\Omega(a)$ and the base line segment AT.

Solution by J.W.Clawson, Collegeville, Pennsylvania. Draw MN and CT perpendicular to AT. Let angle $MCT = \theta$ and $CT = a$. The area required = area AMN + area trapezoid $NMCT$ − area sector MCT.
Now, for M, $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$.

$$\text{Hence area} = a^2\int_0^\theta (1-\cos\theta)^2\,d\theta + \frac{a^2\sin\theta}{2}(2-\cos\theta) - \frac{a^2\theta}{2}$$

$$= 3/2\,a^2\theta - 2a^2\sin\theta + (a^2/2)\sin\theta\cos\theta + a^2\sin\theta - (a^2/2)\sin\theta\cos\theta$$

$$= a^2(\theta - \sin\theta)$$

$$= ax.$$

Also solved by Stanley P. Franklin, Memphis State University; Joseph D.E.Konhauser, State College, Pennsylvania; Arne Pleijel, Trollhattan, Sweden and the proposer.

Solution to Proposal 372:
***Mathematics Magazine*, 33, (1959-1960), 112.**

A Trigonometric Identity

372. [March 1959] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Prove the identity

$$\sin^2(\theta_1+\theta_2+\cdots+\theta_n) = \sin^2\theta_1 + \cdots + \sin^2\theta_n + 2\sum_{1\leq i<j\leq n}^{n} \sin\theta_i \sin\theta_j \cos(\theta_1+2\theta_{i+1}+\cdots+2\theta_{j-1}+\theta_j).$$

Solution by the proposer. We proceed by induction. The equality holds for $n = 1$ and $n = 2$. Let the property be true for $n = p$. Then setting

$$\theta = \theta_1 + \cdots + \theta_p$$

it will suffice to prove the equality obtained by subtraction

$$\sin^2(\theta+\theta_{p-1}) - \sin^2\theta = \sin^2\theta_{p-1} - 2\sum_{i=1}^{p}\sin\theta_1\sin\theta_{p+1}\cdot\cos(\theta_i+2\theta_{i+1}+\cdots+2\theta_p+\theta_{p+1}).$$

The left hand side, A, is seen to be equal to

$$A = \sin\theta_{p+1}\sin(2\theta+\theta_{p+1})$$

The right hand side, B, is equal to

$$B = \sin^2\theta_{p+1} + \sin\theta_{p+1}\sum_{i=1}^{p} 2\sin\theta_1\cos(\theta_i+2\theta_{i+1}+\cdots+2\theta_p+\theta_{p+1})$$

$$= \sin^2\theta_{p+1} + \sin\theta_{p+1}\sum_{i=1}^{p}[\sin(2\theta_i+2\theta_{i+1}+\cdots+2\theta_p+\theta_{p+1})$$

$$- \sin(2\theta_{i+1}+\cdots+2\theta_p+\theta_{p+1})]$$

$$= \sin^2\theta_{p+1} + \sin\theta_{p+1}[(\sin(2\theta_1+2\theta_2+\cdots+2\theta_p+\theta_{p+1}) - \sin\theta_{p+1}]$$

$$= \sin\theta_{p-1}\cdot\sin(2\theta_1+\cdots+2\theta_p+\theta_{p+1}) = A$$

The equality $A = B$ proves that the equality holds for $n = p+1$. The result follows by induction.

Solution to Proposal 380:
Mathematics Magazine, **33, (1959-1960), 172.**

A System of Equations

380. [May 1959] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Solve the system of equations

$$x(z-a)+u(x+u) = 0 \tag{1}$$

$$y(x-b)+u(y+u) = 0 \tag{2}$$

$$z(y-c)+u(z+u) = 0 \tag{3}$$

where $abc \neq 0$ and $a^{-1}+b^{-1}+c^{-1} = u^{-1}$.

Solution by Chih-yi Wang, University of Minnesota. By performing the operations multiply (1) by y, multiply (2) by z, multiply (3) by x and applying (2), (3), (1) respectively we get

$$xyz-aby+auy+au^2+buy-u^3 = 0 \tag{4}$$

$$xyz-bcz+buz+bu^2+cuz-u^3 = 0 \tag{5}$$

$$xyz-cax+cux+cu^2+aux-u^3 = 0 \tag{6}$$

By performing the operations (4) – (5), (5) – (6), (6) – (4) we get

$$(au+bu-ab)y+(bc-bu-cu)z = (b-a)u^2 \tag{7}$$

$$(ca-cu-au)x+(cu+bu-bc)z = (c-b)u^2 \tag{8}$$

$$(au+cu-ca)x+(ab-au-by)y = (a-c)u^2 \tag{9}$$

Since the augmented matrix of (7), (8), (9) is of rank 2, we can calculate two variables in terms of the third, so we get

$$y = \frac{c^2}{a^2}z - \frac{c(b-a)}{ab}u \tag{10}$$

$$x = \frac{b^2}{a^2}z + \frac{b(c-b)}{ca}u \tag{11}$$

By substituting (10) into (3) we get, after simplification,

$$\left(\frac{c}{a}z-u\right)^2 = 0$$

whence by aid of (10) and (11), we obtain

$$x = (b/a)\,u, \quad y = (c/b)\,u, \quad z = (a/c)\,u.$$

Note that we have used the relations $abc \neq 0$, $a^{-1}+b^{-1}+c^{-1} = u^{-1}$ whenever necessary.

Also solved by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; Victor Ch'in, Kent State University, Kent, Ohio; Melvin Hochster, Stuyvesant High School, New York; and the proposer.

Solution to Proposal 384:
***Mathematics Magazine*, 33, (1959-1960), 230.**

An Infinite Group

384. [September 1959] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Let (a_{ij}) be a matrix of nth order the sum of the elements of whose rows equals 1. Prove that the totality $[(a_{ij})]$ form a group of infinite order.

Solution by D. A. Breault, Sylvania Electric Products, Inc. We assume that the proposed group operation is multiplication, and that the sum condition means that

$$[1] \qquad \sum_{j=1}^{n} a_{ij} = 1 \quad \text{for} \quad i = 1, 2, \cdots, n$$

The system has

(1) *Closure:* for if $A = [a_{ij}]$, and $B = [b_{ij}]$, we have

$$C_{ij} = (AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} ;$$

whence

$$\sum_{j=1}^{n} c_{ij} = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{n} a_{ik} \left(\sum_{j=1}^{n} b_{kj} \right) = \sum_{k=1}^{n} a_{ik} = 1$$

for each i.

(2) *Associativity:* which can be demonstrated by the use of summations similar to the above.

(3) *Identity:* The usual identity matrix $I = \delta_{ij}$ serves here also.

(4) *Inverses:* given a matrix A, which satisfies [1], it can be shown that A^{-1} satisfies [1] also, *whenever it exists*! Hence the totality of *non-singular* matrices satisfying [1] form a group, but not the unrestricted set.

Also solved by the proposer.

Solution to Proposal 398:
Mathematics Magazine, 34, (1960-1961), 51.

Simultaneous Quadratics

398. [January 1960] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Determine the roots of the equations

$$x^2 + y_1 x + y_2 = 0$$

$$y^2 + x_1 y + x_2 = 0$$

where the coefficients (real numbers) in one equation are the roots of the other.

Solution by Harry M. Gehman, University of Buffalo.

The relations between roots and coefficients give these four equations:

$$x_1 + x_2 = -y_1$$

$$x_1 x_2 = y_2$$

$$y_1 + y_2 = -x_1$$

$$y_1 y_2 = x_2$$

From the first and third equations, $x_2 = y_2$.

Case I. If $x_2 = y_2 = 0$, then $x_1 = -y_1 = a$, where a is arbitrary, the equations are

$$x^2 - ax = 0 \quad \text{and} \quad x^2 + ax = 0$$

whose roots are a, 0 and $-a$, 0 respectively.

Case II. If $x_2 = y_2 \neq 0$, then $x_1 = y_1 = 1$, and $x_2 = y_2 = -2$. Both equations become

$$x^2 + x - 2 = 0$$

whose roots are 1, -2. Note that there is no need for the condition that the coefficients be real.

Also solved by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; Sidney Kravitz, Dover, New Jersey; A. J. Kokar, School of Mines, Adelaide, Australia; Rostyslaw Lewyckyj, University of Toronto; Ernest E. Moyers, University of Mississippi; F. D. Parker, University of Alaska; Charles F. Pinzka, University of Cincinnati; Arne Pleijel, Trollhattan, Sweden; Robert E. Shafer, University of California Radiation Laboratory; C. M. Sidlo, Framingham, Massachusetts; William Squire, Southwestern Research Institute, San Antonio, Texas; Harvey Walden, Rensselaer Polytechnic Institute; Chih-yi Wang, University of Minnesota; Dale Woods, Northeastern Missouri State College; and the proposer.

Solution to Proposal 407:
***Mathematics Magazine*, 34, (1960-1961), 115.**

Resistance In A Cube

407 [March 1960] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

The twelve edges of a cube are made of wires of one ohm resistance each. The cube is inserted into an electrical circuit by:

a) two adjacent vertices,
b) two opposite vertices of a face,
c) two opposite vertices of the cube.

Which offers the least resistance?

Solution by C. W. Trigg, Los Angeles City College.

It may be inferred that the least resistance occurs in (a) since there is a single-edge connector between the terminals. For confirmation:

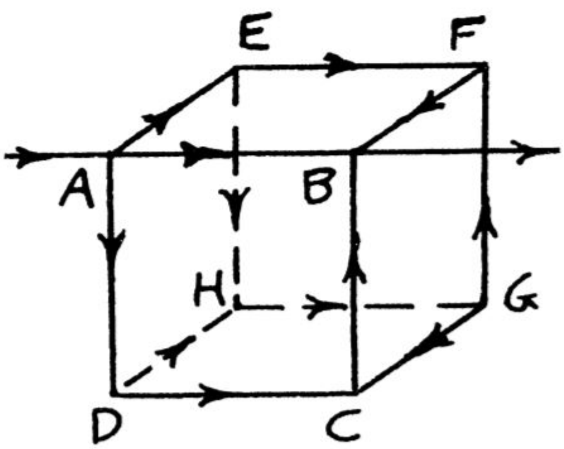
In the figures, the direction of current flow is shown in each case. Below each cube a schematic diagram is shown wherein corners at the same potential, as determined by symmetry, are represented by the same point. Each situation is thus reduced to the simple case of repeated application of the laws of parallel circuits. So:

A) $$1/R = 1/r + 1/\{r/2 + 1/[2/r + 1/(r/2 + r + r/2)] + r/2\},$$

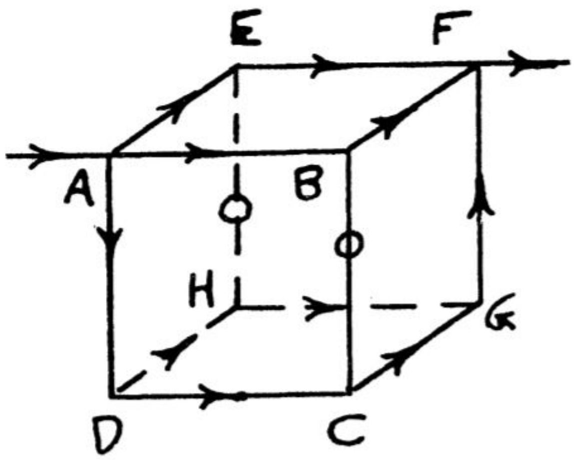
whence $R = 7/12\, r$, where R is the resistance of the cube and r is 1 ohm.

B) E, B, H and C are at the same potential, so

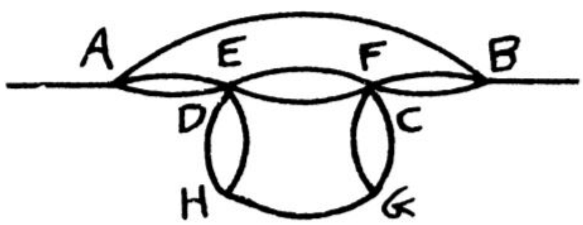
$$R = 2/\{1/r + 1/r + 1/[r + r/2]\} \quad \text{or} \quad 3r/4 .$$

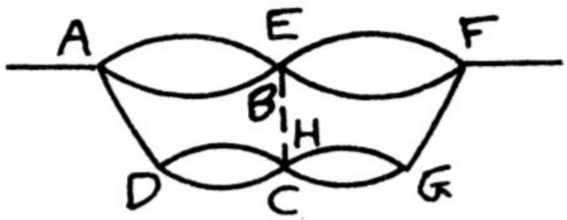


(a)



(b)





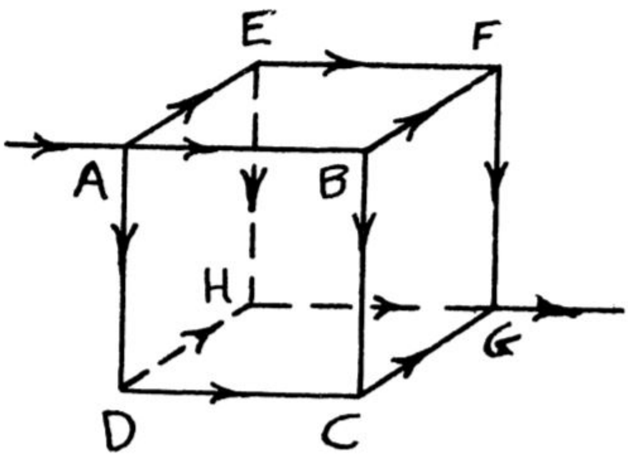
C) $R = r/3 + r/6 + r/3$

or $5r/6$.

Cases (a) and (c) are solved on pages 277-279 of *Magnetism and Electricity* by E. E. Brooks and A. W. Poyser, Longmans, Green and Co. (1920).

Case (c) is Quickie 32, MATHEMATICS MAGAZINE, March 1951, November 1959.

Also solved by Charles F. Pinzka, University of Cincinnati; and the proposer (partially). One incorrect solution was received.



(c)

Solution to Proposal 415:
***Mathematics Magazine*, 34, (1961), 178.**

A Trigonometric Sum

415. [May 1960] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*
Prove

$$\sum_{p=0}^{n} \binom{n}{p} \cos(p)x \sin(n-p)x = 2^{n-1} \sin nx .$$

Solution by Josef Andersson, Vaxholm, Sweden. (Translated and paraphrased by the editor.)

Making use of the formulas

$$\sum_{p=0}^{n} \binom{n}{p} = 2^n \qquad \text{and} \qquad \binom{n}{n-p} = \binom{n}{p} ,$$

the original sum can be written

$$\frac{1}{2}\sum_{p=0}^{n} \binom{n}{p} \sin nx + \frac{1}{2}\sum_{p=0}^{n} \binom{n}{p} \sin(n-2p)x = 2^{n-1} \sin nx + \frac{s}{2} .$$

It remains to be proven that $s = 0$. Now from the substitution $p = n - p'$ it follows that

$$s = \sum_{p'=n}^{0} \binom{n}{n-p'} \sin(2p'-n) = -s .$$

Therefore $s = 0$.

Also solved by J. L. Brown, Ordance Research Laboratory, Pennsylvania State University; L. Carlitz, Duke University; James C. Ferguson, Lynnwood, Washington; A. F. Hordam, University of New England, Armidale, NSW, Australia; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; William Squire, Southwest Research Institute, San Antonio, Texas; Chih-Yi Wang, University of Minnesota, and the proposer.

Comment on Proposal 415:
Mathematics Magazine, 34, (1961), 308.

Comment on Problem 415

415. [May 1960, January 1961] *Proposed by Huseyin Demiř, Kandilli, Eregli, Kdz., Turkey.*

Prove

$$\sum_{p=0}^{n} \binom{n}{p} \cos(p)x \sin(n-p)x = 2^{n-1} \sin nx \ .$$

Comment by Louis Brand, University of Houston.

In the problem of a trigonometric sum a much simpler solution is as follows: Call the sum S and make the index change $p = n-q$; adding the two sums now gives

$$2S = \sum_{p=0}^{n} \binom{n}{p} \sin nx = 2^{n} \sin nx \ .$$

Solution to Proposal 419:
Mathematics Magazine, 34, (1961), 239.

Constant Speed Curve

419. [September 1960] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Determine the path in a vertical plane such that when a particle moved, under gravity, with an initial velocity v_0 from a point of the path, the particle maintained a constant speed along the path. Assume no friction.

Solution by the proposer.

Let Ox, Oy be the axes of coordinates taken in the vertical plane such that Oy points downward and Ox to the left. Let the particle be dropped from O. It reaches the velocity v_0 at a point A of Oy with $y_0 = OA = \frac{v_0^2}{2g}$.

Since there is no friction, the velocity along the path is the projection of the velocity $v = \sqrt{2gy}$, and we write $v_0 = v \cos \alpha$ where

$$v_0 = \sqrt{2gy_0}\,, \quad v = \sqrt{2gy}\,, \quad \cos^2\alpha = \frac{1}{(1+tg^2\alpha)} = \frac{1}{(1+y'^2)}$$

and get $y_0 = y/(1+y'^2)$.

The variables separate and give

$$x = \int_{y_0}^{y} \frac{dy}{\sqrt{(y-y_0)/y_0}} = \tfrac{1}{2}\sqrt{y_0}\sqrt{y-y_0}$$

$$y = \frac{4x^2}{y_0} + y_0$$

$$y = \frac{8g}{v_0^2}x^2 + \frac{v_0^2}{2g}\,.$$

The path is a parabola tangent to Oy at A, Oy being the tangent at the vertex.

Solution to Proposal 425:
Mathematics Magazine, 34, (1961), 300.

Euler's Phi-function

425. [November 1960] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

If $n-1$ and $n+1$ are twin prime numbers, prove that $3\phi(n) \leqq n$ where ϕ denotes Euler's ϕ-function.

I. *Solution by Dermott A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts.*

If $n+1$ and $n-1$ are prime, then n is both even and a multiple of 3, so that for some m, $n = 6m$, and we have:

$$\phi(n) = \phi(6)\,\phi(m) = 2\,\phi(m)\,,$$

while

$$\phi(m) = m \prod_{p\,|\,m} (1 - \frac{1}{p})\,;$$

so

$$3\,\phi(n) = 6\,\phi(m) = 6m \prod_{p\,|\,m} (1 - \frac{1}{p})\,,$$

but $6m = n$ so,

$$3\,\phi(n) = n \prod_{p \mid m} \left(1 - \frac{1}{p}\right) ;$$

whence

$$3\,\phi(n) \leqq n ,$$

as required.

II. *Solution by L. Carlitz, Duke University.*

It is evidently necessary to assume $n > 4$. Since $n-1$ and $n+1$ are primes and $n > 4$ it follows that n is divisible by 3. Also n must be even so that n is divisible by 6. We shall now show that if

(1) $$n = 2^{\alpha} 3^{\beta} m \quad (\alpha \geqq 1,\ \beta \geqq 1,\ (m, 6) = 1) ,$$

then

$$\phi(n) \leqq \frac{n}{3} .$$

Indeed from (1)

(2) $$\phi(n) = 2^{\alpha} 3^{\beta-1} \phi(m) \leqq 2^{\alpha} 3^{\beta-1} m = \frac{1}{3n} .$$

Remark: It is not difficult to show that

(3) $$\phi(n) = \frac{n}{3}$$

if and only if

(4) $$n = 2^{\alpha} 3^{\beta} , \quad (\alpha \geqq 1,\ \beta \geqq 1) .$$

We have seen above that (4) implies (3). Now if (3) holds it is clear that n is divisible by 3. Put $n = 3^{\alpha} k$, where $\alpha \geqq 1$; then (3) becomes

$$2 \cdot 3^{\alpha} \phi(k) = n ,$$

so that n is even. Now put

$$n = 2^{\alpha} 3^{\beta} m \quad (\alpha \geqq 1,\ \beta \geqq 1,\ (m, 6) = 1) .$$

Then if $m > 1$ it follows from (2) that

$$\phi(n) < \frac{n}{3} .$$

This completes the proof of the equivalence of (3) and (4).

Also solved by Brother Alfred, St. Mary's College, California; Leon Bankoff, Los Angeles, California; Maxey Brooke, Sweeney, Texas; B. A. Hausman, S. J., West Baden College, Indiana; Vern Hoggatt, San Jose State College; Sidney Kravitz, Dover, New Jersey; D. L. Silverman, Fort Meade, Maryland; Dale Woods, Northeast Missouri State Teachers College; and the proposer.

Comment on Proposal 425:
Mathematics Magazine, 34, (1961), 433.

Comment on Problem 425

425. [November 1960 and May 1961] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

If $n-1$ and $n+1$ are twin prime numbers, prove that $3\phi(n) \leqq n$ where ϕ denotes Euler's ϕ-function.

Comment by David A. Klarner, Napa, California.

The solution given by Dermott A. Breault contains an error. In the proof we find the statement, "If $n+1$ and $n-1$ are prime, n is even and a multiple of 3, so that for some m, $n = 6m$, and we have

$$\phi(n) = \phi(6)\phi(m) = 2\phi(m) \text{ .''}$$

This is only true when $(6, m) = 1$. In fact, the twin primes 11, 13 yield

$$\phi(12) = \phi(6) \cdot \phi(2) = 2 ,$$

but $\phi(12) = 4$. Therefore the method of proof given would have to be altered to make it valid.

Comment on Proposal 437:
Mathematics Magazine, 34, (1961), 371.

A Well Known Problem

437. [January 1961] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Prove or disprove the statement: The number of odd coefficients in the binomial expansion of $(a+b)^{[n]}$ is a power of 2, the exponent of 2 being the number of 1's appearing in the expression of n in the binary number system.

Editor's note: Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania, pointed out that a simpler version of this problem appeared as Problem 7, Part II, in the Putnam Competition of 1956. The given problem appeared as Problem E 1288 in the *American Mathematical Monthly* in November 1957 with solution and references given in the May, 1958 issue.

Elementary Problem 1288, *American Mathematical Monthly*, 64, (1957), 671.

E 1288. *Proposed by S. H. Kimball, University of Maine*

The number of odd binomial coefficients in any finite binomial expansion is a power of 2 (Putnam Mathematical Competition, this Monthly [1957, p. 24]). Prove that the power of 2 is the number of 1's in the binary scale expression for n in $(x+y)^n$.

Solution to Problem 1288:
American Mathematical Monthly, 65, (1958), 368.

Odd Binomial Coefficients

E 1288 [1957, 671]. *Proposed by S. H. Kimball, University of Maine*

The number of odd binomial coefficients in any finite binomial expansion is a power of 2 (Putnam Mathematical Competition, this MONTHLY [1957, p. 24]). Prove that the power of 2 is the number of 1's in the binary scale expression for n in $(x+y)^n$.

I. *Solution by T. R. Hatcher and J. A. Riley, Parke Mathematical Laboratories, Carlisle, Mass.*

Let h and n be positive integers with $h<n$. We define the *binary length* of n, $L(n)$, to be the number of ones in the binary representation of n, and the *binary capacity* of n, $C(n)$, to be the exponent of the highest power of two which divides n. We say "h is contained in n," written $h\subset n$, if when h has a one in a certain binary place, n also has a one in the corresponding binary place; that is, the binary representation of h can be obtained from that of n by changing ones to zeros.

The following properties are easily proved:

(1) $C(n)$ is the number of terminating zeros in the binary representation of n.

(2) $C(n)=0$ if and only if n is odd.

(3) $C(ab)=C(a)+C(b)$, $C(a/b)=C(a)-C(b)$.

(4) $L(n)=L(h)+L(n-h)$ if and only if $h\subset n$.

(5) $C(n)=1+L(n-1)-L(n)$.

(6) $C(n!)=n-L(n)$.

(7) $C\binom{n}{h}=L(h)+L(n-h)-L(n)$.

The corollary of the following theorem gives the solution.

THEOREM. *$\binom{n}{h}$ is odd if and only if $h\subset n$.*

Proof. If $\binom{n}{h}$ is odd, $C\binom{n}{h}=0$ and by (7) $L(n)=L(h)+L(n-h)$. Thus, by (4), $h\subset n$. Conversely, if $h\subset n$, then $L(n)=L(h)+L(n-h)$ and $C\binom{n}{h}=0$.

COROLLARY. *The number of integers h such that $\binom{n}{h}$ is odd is $2^{L(n)}$.*

Proof. The number of integers h with $h\subset n$ and $L(h)=j$ is $\binom{L(n)}{j}$. Thus the number of integers h for which $\binom{n}{h}$ is odd is simply

$$\sum_{j=0}^{L(n)}\binom{L(n)}{j} = 2^{L(n)}.$$

II. *Remarks by Leo Moser, University of Alberta*. Problem E 1288 is a special case of 4723 [1957, 116]. The solution of that problem is the following:

If $n=a_0+a_1p+a_2p^2+\cdots+a_kp^k$, $0\leqq a_i<p$, $i=0, 1, 2, \cdots, k$, then the number of solutions of $(\binom{n}{r}, p)=1$, $r=0, 1, \cdots, n$, is $\prod_{i=0}^{k}(a_i+1)$.

The result in E 1288 is contained in J. W. L. Glaisher, "On the residue of a binomial coefficient with respect to a prime modulus," *Quarterly Journal of Mathematics*, vol. 30, 1899, pp. 150–156. More recently a proof was given by J. B. Roberts, "On binomial coefficient residues," *Canadian Journal of Mathematics*, vol. 9, 1957, pp. 363–370.

Also solved by D. R. Brillinger, Leonard Carlitz, Joe Lipman, D. C. B. Marsh, Paul Schillo, and the proposer.

Solution to Proposal 440:
Mathematics Magazine, 34, (1961), 427.

Circle Packing

440. [March 1961] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Consider a packing of circles of radius r such that each is tangent to its six surrounding circles. Let a larger circle of radius R be drawn concentric with one of the small circles. If N is the number of small circles contained in the larger circle, prove that

$$N = 1+6n+6\sum_{p=1}^{n}[\tfrac{1}{2}(\sqrt{4n^2-3p^2}-p)]$$

where $n=[\tfrac{1}{2}(\frac{R}{r}-1)]$, the square brackets designating the greatest integer function.

Solution by Alan Sutcliffe, Knottingley, Yorkshire, England.

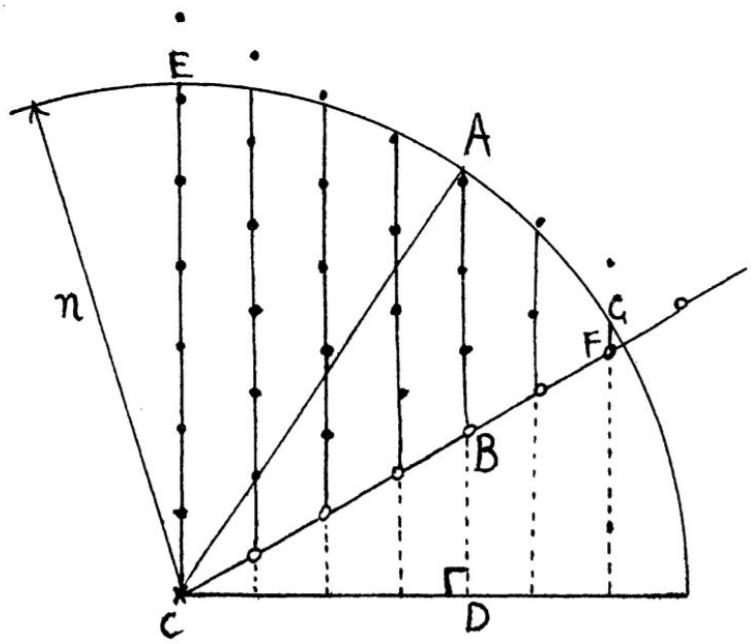
The expression is not quite correct. For example when $\frac{R}{r}=2\sqrt{3}+1$ we have $n=1$ and hence $N=7$, while the correct value is $N=13$. The correct expression is

$$N = 1+6[n]+6\sum_{p=1}^{[n]}[\tfrac{1}{2}(\sqrt{4n^2-3p^2}-p)] = 1+6\sum_{p=0}^{[n]}[\sqrt{n^2-(3/5)p^2}-\frac{p}{2}]\ ,$$

where $n=\tfrac{1}{2}(\frac{R}{r}-1)$.

To prove this we shall first assume unit distance between adjacent centers, and find the number of centers within a circle of radius r. Because of the triangular nature of the array of centers, we need consider only one of the six similar sectors of the circle as shown in the diagram, where the centers marked $\circ$ are in the adjoining sector and the common center C

is in no sector. Clearly the number of centers contained within the sector



is the sum of the integral part of the lengths, such as AB, from CE to FG. Let $CB = p$, which will be an integer. Then, since angle $BCD = 30°$, $CD = (\sqrt{3}/2)p$ and $BD = p/2$. As $AC^2 = AD^2 + CD^2$ we have

$$n^2 = (AB + \frac{p}{2})^2 + \frac{3}{4}p^2 \ .$$

Hence

$$AB = \sqrt{n^2 - (3/4)p^2} - \frac{p}{2} \ .$$

The number of centers within the sector is the sum of the integral part of this from $p = 0$ to $[n]$. Since there are six sectors and the common center C, we have

$$N = 1 + 6 \sum_{p=0}^{[n]} [\sqrt{n^2 - (3/4)p^2} - \frac{p}{2}] \ .$$

Now in fact the centers are not unit distance, but $2r$ apart. So that a radius $R = 2rn$ will contain N centers. Thus a radius $R = 2rn + r$ will contain N circles, giving $n = \frac{1}{2}(\frac{R}{r} - 1)$, which completes the proof.

Comment on Proposal 440:
Mathematics Magazine, 35, (1962), 316.

440. [March and November 1961]. *Comment by Huseyin Demir, Middle East Technical University, Akara, Turkey.*

The number N given in the statement is correct. N denotes the number of small circles contained entirely by the larger circle (tangency being included). The number N offered by A. Sutcliffe includes also the partly contained circles and therefore both numbers are correct.

Solution to Proposal 458:
Mathematics Magazine, 35, (1962), 126.

De Moivre's Theorem

458. [September 1961] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

A student used DeMoivre's theorem incorrectly as

$$(\sin \alpha + i \cos \alpha)^n = \sin n\alpha + i \cos n\alpha \; .$$

For what values of α does the equation hold for every integer n?

Solution by Dermott A. Breault, Sylvania Applied Research Laboratory, Waltham, Massachusetts. Let

$$z = \cos \theta + i \sin \theta \; .$$

Then using DeMoivre's Theorem correctly we have

$$z^n = \cos n\theta + i \sin n\theta \; .$$

The proposed relation is that $(i/z)^n = (i/z^n)$ which implies that

$$(1/z^n)(i^n - i) = 0 \; .$$

But $z^{-n} \neq 0$, so there are no values of θ for which the proposal holds for every integer n, but it is an identity for all n of the form $n = 4k+1$.

Also solved by Brother U. Alfred, St. Mary's College, California; Leonard Carlitz, Duke University; Alan B. Delfino, St. Mary's College, California; P. D. Goodstein, University of Leicester, England; Harvey H. Green, R. C. A. Ascension Island (partially); Richard Levitt, Boston Latin School; David L. Silverman, Beverly Hills, California; Paul Stygar, Yale University; W. C. Waterhouse, Harvard University; and the proposer.

Solution to Proposal 472:
***Mathematics Magazine*, 35, (1962), 255.**

Symmetric Conics

472 [January 1962]. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Let (C) be a conic and M be a variable point on it. Let T be the point symmetric to M with respect to the main axis, and t the tangent line at T. Denote the intersection of the perpendicular from M to t with the line joining T to the center of the conic by I. If M' is symmetric to M with respect to I, prove that: 1. The locus of M' is another conic (C') of the same kind as (C). 2. The conics (C) and (C') are confocal.

Solution by R. D. H. Jones, College of William and Mary, Virginia. Let the conic be $x^2/a^2+y^2/b^2=1$, let M be $(a\cos\Delta, b\sin\Delta)$ so T is the point $(a\cos\Delta, -b\sin\Delta)$, and t, the tangent at T, is $(x/a)\cos\Delta-(y/b)\sin\Delta=1$. MI is the line through M perpendicular to t and therefore is:

$$v - b\sin\Delta = \frac{-a\sin\Delta}{b\cos\Delta}\cdot(x-a\cos\Delta). \tag{1}$$

The line joining T to the center of conic is

$$\frac{x}{a\cos\Delta}+\frac{y}{b\sin\Delta}=0. \tag{2}$$

The point I is the intersection of (1) and (2) and is found to have coordinates:

$$\frac{a(a^2+b^2)}{a^2-b^2}\cos\Delta, \qquad \frac{-b(a^2+b^2)}{a^2-b^2}\sin\Delta.$$

By hypothesis M' is symmetric to M with respect to I and therefore has coordinates:

$$x_{M'} = 2x_I - x_M = \frac{a^3+3ab^2}{a^2-b^2}\cos\Delta$$

similarly

$$v_{M'} = 2y_I - y_M = \frac{-(3a^2, b+b^3)}{a^2-b^2}\sin\Delta.$$

Let

$$A=\frac{a(a^2+b^2)}{a^2-b^2} \quad\text{and}\quad B=\frac{b(a^2+b^2)}{a^2-b^2}.$$

If a and b are real and a greater than b, then A and B are real and A greater than B. Therefore if C is an ellipse the locus of M' is an ellipse. If, however, b is imaginary B is imaginary: hence if C is an hyperbola, so is the locus of M'. It is readily shown that $A^2-B^2=a^2-b^2$: therefore the locus of M' is confocal with C.

Solution to Proposal 487:
***Mathematics Magazine*, 36, (1963), 76.**

The Square Root of a Matrix

487. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Find the square root of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Solution by Maurice Brisebois, Université de Sherbrooke, Canada.

Let X, U be arbitrary square matrices of order n, A a given non-singular matrix, $\tilde{A}$ a matrix similar to A, $X_{\tilde{A}}$ an arbitrary nonsingular matrix permutable with $\tilde{A}$, (λi) the set of all characteristic values of A; $i=1, \cdots, n$ (they need not be all distinct), E_{p_i} the identity matrix of order p_i with $\sum_i p_i = n$, H_{p_i} the matrix with 1's in the superdiagonal and 0's elsewhere. Let $(\sqrt[n]{\lambda_1 E_{p1} + H_{p1}} \cdots, \sqrt[n]{\lambda_n E_{pn} + H_{pn}})$ be a matrix built with square matrices along the diagonal, matrices of order p_i of the type

$$\begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix} \quad ; i = 1, \cdots, n$$

and having 0's elsewhere.

Then all solutions of the matrix equation

$$X^m = A$$

are given by the formula:

$$X = UX_{\tilde{A}}(\sqrt[m]{\lambda_1 E_{p_1} + H_{p_1}}, \cdots, \sqrt[m]{\lambda_n E_{p_n} + H_{p_n}}) \cdot X_{\tilde{A}}^{-1} U^{-1}.$$

The particular case $m=n=2$ yields:

$$X = UX_{\tilde{A}} \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} X_{\tilde{A}}^{-1} U^{-1}.$$

If the matrix A is singular, a more elaborate study is needed and the existence of the mth roots of A is bound with the existence of a system of admissible elementary divisors for X_2 a matrix such that (X_1, X_2) is a matrix similar to X. (We call a system of elementary divisors for X_2 "admissible" if, after raising X_2 to the mth power, these elementary divisors split and generate the system of elementary divisors for A_2 where $A=(A_1, A_2)$ with A_1 and A_2 similar to X_1 and X_2 respectively.)

Remarks. 1. In the general case, the solutions of $X^m = A(|A| \neq 0)$ are not expressible as polynomials in A unless all λ_i are distinct.

2. The solutions of $X^m = A$ are parametric in nature and the number of parameters present in $X_{\tilde{A}}$ is given by the number N of linearly independent

matrices commuting with A, where $N=\sum_{i+1}^{t}(2i-1)n_i$; $(t\leqq n)$, n_i being the degrees of the non-constant invariant polynomials of A.

3. For some results along this line, see Lusternik-Sobolen, "Elem. of Functional Analysis," p. 283, Dunford-Schwartz, "Linear Operations" (Part I), problem 31 on page 583 and Bellman "Introd. to Matrix Analysis," problems 1–3 on page 93.

Also solved by Brother U. Alfred, St. Mary's College, California; J. A. H. Hunter, Toronto, Canada; Francis D. Parker, University of Alaska; Gilbert Labelle, University of Montreal, Canada; C. F. Pinzka, University of Cincinnati; J. L. Stearn, Washington, D. C.; and the proposer.

Solution to Proposal 498:
Mathematics Magazine, 36, (1963), 201.

A Property of Multiplicative Functions

498. [November 1962] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

If m and n are integers and δ, D are their g.c.d. and l.c.m. respectively, and $d(n)$ denotes the number of divisors of n, $\phi(n)$ being the Euler function, prove that:

$$d(m)d(n) = d(\delta)d(D) \tag{1}$$

$$\phi(m)\phi(n) = \phi(\delta)\phi(D) \tag{2}$$

Solution by L. Carlitz, Duke University.

The result is a special case of the following theorem. Let $f(n)$ be an arbitrary factorable function, that is

$$f(mn) = f(m)f(n)$$

for all m, n such that $(m, n) = 1$. Then

$$(*) \qquad f(m)f(n) = f(\delta)f(D),$$

where $\delta = (m, n)$ and $D = [m, n]$, the greatest common divisor and the least common multiple, respectively.

The proof of (*) is immediate. If

$$m = \Pi p^r, \qquad n = \Pi p^s,$$

then

$$\delta = \Pi p^{r'}, \qquad D = \Pi^{s'},$$

where $r' = \min(r, s)$, $s' = \max(r, s)$. Since $r' + s' = r + s$, we have

$$f(\delta)f(D) = \Pi p^{r'+s'} = \Pi p^{r+s} = f(m)f(n).$$

Also solved by Stephen R. Cavior, Duke University; Daniel I. A. Cohen, Brooklyn, New York; George Diderick, University of Wisconsin; Murray S. Klamkin, State University of New York at Buffalo; David A. Klarner, Humboldt State College, California; Gilbert Labelle, Université de Montréal; Jerry L. Pietenpol, Columbia University; Robert Prielipp, University of Wisconsin; Sam Sesskin, Hempstead, New York; David L. Silverman, Beverly Hills, California; Irene Williams, Converse College, South Carolina; and the proposer.

Solution to Proposal 509:
***Mathematics Magazine*, 36, (1963), 321.**

An American Alphametic

509. [March 1963] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the cryptarithm

$$\begin{array}{r} U\ N\ I\ T\ E\ D \\ S\ T\ A\ T\ E\ S \\ \hline A\ M\ E\ R\ I\ C\ A \end{array}$$

in the base 11, introducing the digit α.

Solution by Anton Glaser, Ogontz Campus, Pennsylvania State University.

My solution was obtained as follows:

(1) $A=1$ [In any numeration system, adding two digits can result in "carrying" at most unity.]

(2) $D\neq 0$ [Suppose $D=0$, then we get contradiction of $S=A=1$ vs. $S\neq A$]

(3) $S\neq 0$ [Similar to (2)]

(4) $E\neq 0$ [Suppose $E=0$, then $C=A=1$ contradicting $C\neq A$]

(5) $T\neq 0$ [Suppose $T=0$, then either $I=A=1$ vs. $I\neq A$ or $T=I=0$ vs. $T\neq I$]

(6) $U\neq 0$ and $S\neq 0$ by usual rules of cryptarithms

(7) $E\neq\alpha$ [Suppose $E=\alpha$, then $C=E=\alpha$ vs. $C\neq E$]

(8) $U+S>9$

(9) $U+S>\alpha$ if nothing was "carried" from previous column

(10) $D+S=11_{(\text{eleven})}=12_{\text{ten}}=\text{twelve}$

Only the digits shown in table at right are possible for D and S, and only in the combinations shown.

[Since $A=1$ neither D nor S can be 1]

[Neither D nor S can be 6, since either would imply $D=S=6$ vs. $D\neq S$]

D	
2	α
3	9
4	8
5	7
7	5
8	4
9	3
α	2

(11) $T\neq 2$, $T\neq 3$, $T\neq 4$, and $T\neq 5$ [For $T=2$, $T=3$, $T=4$, and $T=5$ and the seven possible values of E that go with each of these four values of T, there resulted in each case a contradiction of some sort.]

(12) For $T=6$ and $E=5$, the remaining letters could be assigned a one-to-one correspondence with the remaining digits that would satisfy the cryptarithm.

$$\begin{array}{r} 8\ \alpha\ 2\ 6\ 5\ 3 \\ 9\ 6\ 1\ 6\ 5\ 9 \\ \hline 1\ 7\ 5\ 4\ 2\ 0\ 1 \end{array}$$

Also solved by Josef Andersson, Vaxholm, Sweden; Merrill Barneby, University of North Dakota; Maxey Brooke, Sweeny, Texas; Harry M. Gehman, State University of New York at Buffalo; Wahin Ng, San Francisco, California; Norman Harelik, Mather High School, Chicago, Illinois; J. A. H. Hunter, Toronto, Ontario, Canada; Robert Sandling, Columbia University; Anita Skelton, Watervliet Arsenal, New York; David L. Silverman, Beverly Hills, California; Orvan Sommers, West Bend High School, Wisconsin; C. W. Trigg, Los Angeles City College; Hazel S. Wilson, Jacksonville University, Florida; Brother Louis F. Zirkel, Archbishop Molloy High School, Jamaica, New York; and the proposer.

Solution to Proposal 517:
Mathematics Magazine, 37, (1964), 56.

Parabolic Areas

517. [May 1963] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let F and d be the focus and directrix of a parabola. If M and N are any two points on the parabola and M', N' are their respective projections on d, show that

$$\frac{\text{Area } FMN}{\text{Area } N'M'MN} = \text{Constant.}$$

I. *Solution by Francis D. Parker, University of Alaska.*

Using a focal length of F and orienting the directrix on the x-axis and the focus on the y-axis, we may use $y=x^2/4F+F$ as the equation of the parabola. If the abscissas of M and N are a and b, respectively, straightforward calculations yield

$$\text{Area } MM'N'N = \int_a^b y dx = \frac{b-a}{12F}[12F^2 + a^2 + ab + b^2]$$

and

$$\text{Area } FMN = \frac{b-a}{24F}[12F^2 + a^2 + ab + b^2].$$

Hence, the ratio of the areas is independent of F, a, and b, and is equal to **1/2**.

II. *Solution by Joel Kugelmass, Stanford University and the National Bureau of Standards.*

It is clear that any parabola $f_1(x)$ can be transformed into another parabola $f_2(x)$ by applying a projective transformation P, an orthogonal transformation O and suitable rotations and translations. All of these transformations preserve the ratio of the area of the triangular region to that of the trapezoidal region. Hence we may transform any parabola to $y=x^2$. If we transform again so that $\lim M-N=0$, the areas clearly approach the length of their altitudes which in turn approaches p, the distance from the focus to the center. Now the function $z=(p+\epsilon_1)/(p+\epsilon_2)$, where the divisor and dividend are the areas of the regions, is monotone after a sufficient number of transformations $(\epsilon_1+\epsilon_2<\delta)$ and hence approaches the limit. Now as all of the ratios are the same under the given transformations, the original ratio equals a constant and the theorem is proved.

Also solved by Josef Andersson, Vaxholm, Sweden; Michael J. Pascual, Watervliet Arsenal, New York; Hazel S. Wilson, Jacksonville University, Florida; and the proposer.

Dermott A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; P. R. Nolan, Department of Education, Dublin, Ireland; and Brother Louis F. Zirkel, Archbishop Molloy High School, Jamaica, New York; each pointed out that the proposal is incorrect if the figures FMN and N'M'MN are considered to be the rectilinear areas instead of areas bounded by the arc of the parabola, MN.

One incorrect solution was received.

Comment on Proposal 517:
Mathematics Magazine, 37, (1964), 517.

Comment on Problem 517

517. [May 1963 and January 1964]. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Comment by Josef Andersson, Vaxholm, Sweden.

This property of the areas characterizes in a way the parabola. If, in fact, $r=f(\phi)$ is the equation of a curve K in polar coordinates, associated to orthonormal coordinates $X'OX$, $Y'OY$ and that for each point P of K we construct PP' equivalent to $OQ(r, 0)$ the point P' traces the curve K'. Let A_1 and A_2

represent the areas between $X'OX$, OP, K and K, PP', K', $X'OX$ respectively. The condition

$$(1) \qquad \frac{1}{2}r^2 = \frac{dA_1}{d\phi} = \frac{1}{2}\cdot\frac{dA_2}{d\phi} = \frac{1}{2}\cdot\frac{dA_2}{d(r\sin\phi)}\cdot\frac{d(r\sin\phi)}{d\phi} = \frac{1}{2}r\frac{d(r\sin\phi)}{d\phi}$$

gives

$$\frac{d(r\sin\phi)}{r\sin\phi} + \frac{d\left(\cot\dfrac{\phi}{2}\right)}{\cot\dfrac{\phi}{2}} = 0, \qquad 2r\cos^2\frac{\phi}{2} = \text{constant}.$$

Therefore, K is a parabola with O as focus, $X'OX$ as axis and therefore K' is the directrix. The hypothesis and the ratio $\frac{1}{2}$ is deducted immediately from (1) if we take at first one of the points at the vertex.

Solution to Proposal 529:
Mathematics Magazine, 37, (1964), 124.

Center of Curvature

529. [September 1963] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

A cycloid (cardioid) rolls on a straight line without sliding. Prove that the locus of the center of curvature of the curve at the point of tangency is a circle (ellipse).

Solution by P. R. Nolan, Department of Education, Dublin, Ireland.

Cycloid. Taking the regular case and putting $\omega t = \alpha$ we have

$$x = a(\alpha - \sin\alpha), \qquad y = a(1 - \cos\alpha).$$

By the usual methods, the arc length from the origin is given by

$$\text{(i)} \qquad S_\alpha = 4a(1 - \cos\alpha/2)$$

and the radius of curvature by

$$\text{(ii)} \qquad P_\alpha = 4a\sin\alpha/2 \cdot$$

Now if one arch of the cycloid rolls once along the y axis, the coordinates of the center of curvature at the point of tangency will be (P_α, S_α). Therefore from (i) and (ii), its locus is

$$x^2 + (y - 4a)^2 = (4a)^2$$

which is a *semicircle*, negative values of $x(P)$ not being admissible, unless we consider the next arch to roll back along the y axis to complete the locus-circle.

Cardioid. In polar coordinates

$$r = a(1 - \cos\theta).$$

As before, this gives

(i) $$S_\theta = 4a(1 - \cos\theta/2)$$

and

(ii) $$P_\theta = (4a/3)\sin\theta/2.$$

Now if the cardioid rolls once along the upper edge of the x axis, the coordinates of the center of curvature at the point of tangency will be (S_θ, P_θ). Therefore from (i) and (ii), its locus is

$$(x - 4a)^2 + 9y^2 = (4a)^2$$

which is the upper *half of an ellipse*, negative values of $y(P)$ not being admissible, unless the same cardioid is also rolled along the lower edge of the axis.

Also solved by the proposer.

Solution to Proposal 537:
Mathematics Magazine, 37, (1964), 277.

Extreme Overlap

537. [January, 1964]. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Determine the relative positions of an equilateral triangle and a square inscribed in the same circle so that their common area shall be an extremum.

Solution by Michael Goldberg, Washington, D. C.

The extrema are the symmetric relative positions of the square and the triangle.

The minimum overlap occurs when a side of the square is parallel to a side of the triangle as shown in Figure 1. The protruding portions of the triangle are marked A and B. For an infinitesimal rotation from this position, an increase in one of the B areas is compensated by the corresponding decrease in the other B, while the area A is reduced.

The maximum overlap occurs when a vertex of the square coincides with a vertex of the triangle as shown in Figure 2. The equal protruding portions of the triangle are marked C. For an infinitesimal rotation, an increase in one C is compensated by a corresponding decrease in the other C, while a portion of the triangle at the third vertex will now protrude.

If the radius of the circle is unity, the areas are given as follows:

$$A = (2 - \sqrt{2})/2$$
$$2B = \sqrt{3}(\sqrt{3} - \sqrt{2})^2/2$$
$$2C = (9 - 5\sqrt{3})/4.$$

Hence,

$$A + 2B = 0.0933, \text{ maximum protrusion, minimum overlap,}$$

$$2C = 0.0849, \text{ minimum protrusion, maximum overlap.}$$

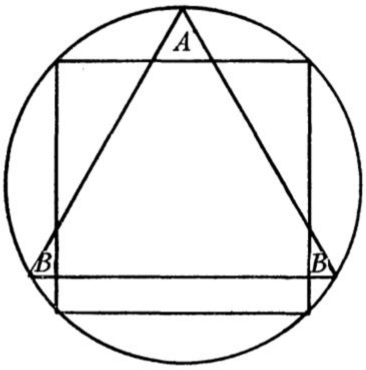


FIG. 1.

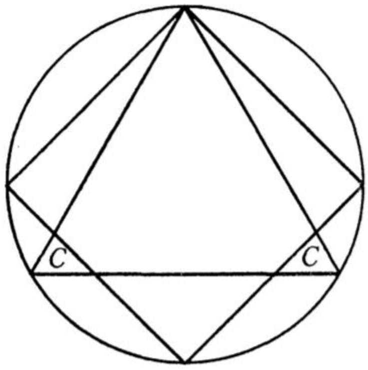


FIG. 2.

Solution to Proposal 544:
***Mathematics Magazine*, 37, (1964), 354.**

A Conditional Alphametic

544. [March, 1964] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the cryptarithm (alphametic)

$$ONE + TWO + SIX = NINE$$

in the base 10, with the following conditions:

a) $ONE < TWO < SIX$

b) $2|TWO$, $6|SIX$, $9|NINE$ where $a|b$ means "a divides b."

Solution by Sister Mary Joy, Notre Dame College, St. Louis, Missouri.

Since each letter represents a different digit, it can readily be seen from condition (a) that $O < T < S$, $S \geqq O+2$, $T \geqq O+1$, and from condition (b) that TWO and SIX are both even.

Observe that E occupies the unit's place in the sum. Thus, $O+X$ must be 10. Both X and O are single digits, neither can be zero, nor can the sum be greater than $1E$. From the fact that N occupies the ten's place in the sum, it follows that $W+I=9$. Also, $W+I \neq 10$ as there is 1 ten carried from the unit's column.

Thus there are four possible ordered pairs for X and O: (8,2), (2,8), (6,4) and (4,6). Now pairs of addends for W and I are chosen such that neither addend duplicates a digit already taken. Possibilities for S are then chosen such that $6|SIX$, where $S \geqq O+2$. If no duplication has occurred thus far, T is chosen so that $O+T+S=NI$. With still no duplication of digits, E is determined such that $9|NINE$.

Consequently the solution is found to be

ONE	217
TWO	392
SIX	408
$NINE$	1017

Also solved by Josef Andersson, Vaxholm, Sweden; Merrill Barneby, University of North Dakota; Maxey Brooke, Sweeny, Texas; J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania; David M. Cohen, East Midwood Day School, Brooklyn, New York; Martin J. Cohen, Beverly Hills, California; Michael P. Cozmanoff, Lew Wallace High School, Gary, Indiana; John A. Dossy, Illinois State University, Normal, Illinois; Joseph M. Fine, Massachusetts Institute of Technology; C. E. Franti, Berkeley, California; Philip Fung, Fenn College, Ohio; Harry M. Gehman, SUNY at Buffalo, New York; Murray Geller, Jet Propulsion Laboratory, Pasadena, California; Anton Glasser, Pennsylvania State University, Abington, Pennsylvania; Garold F. Gregory, Forest Disease Research Laboratory, Delaware, Ohio; C. T. Haskell, California State Polytechnic College, San Luis Obispo, California; Burton S. Holland, Harpur College, New York; William R. Holt, Delaware, Ohio; J. A. H. Hunter, Toronto, Ontario, Canada; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; Janice Langan, Lew Wallace High School, Gary, Indiana; John W. Milsom, Texas A and I, Kingsville, Texas; Wa Hin Ng, San Francisco, California; C. C. Rice, IBM, Endicott, New York; Perry A. Scheinok, Hahnemann Medical College, Philadelphia, Pennsylvania; C. W. Trigg, San Diego, California; A. M. Vaidya, Pennsylvania State University; J. S. Vigder, Ottawa, Canada; Thomas Wojtan, Lew Wallace High School, Gary, Indiana; Dale Woods, Northeast Missouri State TeachersCollege; Charles Ziegenfus, Madison College, Virginia; and the proposer.

Solution to Proposal 563:
Mathematics Magazine, 38, (1965), 122.

Angles in a Hexagon

563. [September, 1964] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let A, B', A', B be four consecutive vertices of a regular hexagon. If M is an arbitrary point of the circumcircle (in particular on arc $A'B'$) and MA, MB intersect BB' and AA' in the points E and F respectively, then prove that:

$$\text{(a)} \quad \sphericalangle MEF = 3\sphericalangle MAF$$

$$\text{(b)} \quad \sphericalangle MFE = 3\sphericalangle MBE.$$

Solution by Richard A. Jacobson, South Dakota State University.

Let $AB'=x$ and $\sphericalangle MAF=a$. Noting that $\sphericalangle AB'B=\sphericalangle AA'B=\sphericalangle AMB=90°$, we have from triangles AMB, $AB'E$ and $AA'B$ that $AM=2x\cos(30+a)$, $BM=2x\sin(30+a)$, $AE=x/\cos(30-a)$ and $BF=x/\cos a$. Thus in triangle EMF we find that

$$\tan(\sphericalangle MEF) = \frac{MF}{ME} = \frac{BM-BF}{AM-AE} = \frac{2x\sin(30+a) - \dfrac{x}{\cos(a)}}{2x\cos(30+a) - \dfrac{x}{\cos(30-a)}}$$

$$= \frac{\dfrac{2\sin(30+a)\cos(a)-1}{\cos a}}{\dfrac{2\cos(30+a)\cos(30-a)-1}{\cos(30-a)}}$$

$$= \frac{2\sin(30+2a)-1}{\cos(a)} \cdot \frac{\cos(30-a)}{2\cos(2a)-1}$$

$$= \frac{2\sin(30+2a)\cos(30-a) - \cos(30-a)}{2\cos(2a)\cos(a)-\cos(a)}$$

$$= \frac{\sin(60+a)+\sin(3a)-\cos(30-a)}{\cos(3a)+\cos(a)-\cos(a)}$$

$$= \frac{\sin(3a)}{\cos(3a)} = \tan(3a).$$

Since $a \leqq 30°$, we have $\sphericalangle MEF = 3\sphericalangle MAF$. Part (b) is done similarly.

Also solved by Leon Bankoff, Los Angeles, California; J. D. E. Konhauser, HRB-Singer, State College, Pennsylvania; Stanley Rabinowitz, Far Rockaway, New York; Sidney Spital, California State Polytechnic College; and the proposer.

Solution to Proposal 572:
Mathematics Magazine, 38, (1965), 242.

A Memorial Cryptarithm

572. [January, 1965] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

To the memory of President Kennedy. Mr. J. F. Kennedy was killed on November 22, 1963. That is, on the day 11-22-1963. Solve the cryptarithm

$$JF\cdot(KEN+NEDY)=(11+22)\cdot 1963$$

in the decimal system.

Solution by Harry M. Gehman, SUNY at Buffalo, New York.

Since $(11+22)\cdot 1963$ is the product of the four primes 3, 11, 13 and 151, the only possible values of JF are 13 and 39. The latter case leads to a contradiction, and hence $J=1$ and $F=3$. From this, it follows that $KEN+NEDY=4983$, which leads to $N=4$, $Y=9$, and either $K=2$, $E=7$, $D=0$ or $K=7$, $E=2$, $D=5$. Thus

$$\begin{aligned}(11+22)\cdot 1963 &= 13\cdot(274+4709)\\ &= 13\cdot(724+4259).\end{aligned}$$

The fact that this problem has two solutions means (to a Republican) that JFK was *not* unique.

Also solved by Robert H. Anglin, Danville, Virginia; Merrill Barneby, University of North Dakota; Murray Berg, Standard Oil Company, San Francisco, California; Charles R. Berndtson, Institute of Naval Studies, Cambridge, Massachusetts; Dermott A. Breault, Sylvania A.R.L., Waltham, Massachusetts; Robert Brodeur, Lachine, Canada; Maxey Brooke, Sweeny, Texas; Allan Chuck, San Francisco, California; R. J. Cormier, Northern Illinois University; Monte Dernham, San Francisco, California; Herta T. Freitag, Roanoke, Virginia; Philip Fung, Fenn College, Ohio; Anton Glaser, Pennsylvania State University, Ogontz Campus; Elmer E. Hunt, Jr., Boise Junior College, Boise, Idaho; J. A. H. Hunter, Toronto, Canada; Joel V. Kamer, Cambridge, Massachusetts; John Koelzer, University of Iowa; Wahin Ng, San Francisco, California; C. C. Oursler, Southern Illinois University (Edwardsville); Harry Panish, Pomona, California; Lawrence A. Ringenberg, Eastern Illinois University; Sidney Spital, California State Polytechnic College; P. K. Subramanian, Miami University, Ohio; Charles W. Trigg, San Diego, California; William K. Viertel, State University Agricultural and Technical College, Canton, New York; Dale Woods, Northeast Missouri State Teachers College; Charles Ziegenfus, Madison College, Virginia; and the proposer.

Solution to Proposal 587:
Mathematics Magazine, 39, (1966), 127.

A Trigonometric Inequality

587. [May, 1965] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Prove the following inequality

$$\left(\frac{\theta + \sin\theta}{\pi}\right)^2 + \cos^4 \frac{1}{2}\theta < 1, \qquad (-\pi < \theta < +\pi).$$

Solution by Samuel Wolf, Linthicum Heights, Maryland.

$$\left(\frac{\theta + \sin\theta}{\pi}\right)^2 + \cos^4 \frac{\theta}{2} = \left(\frac{\theta + \sin\theta}{\pi}\right)^2 + \left(\frac{1 + \cos\theta}{2}\right)^2 = F$$

Differentiating, and setting to zero:

$$\frac{2}{\pi^2}(\theta + \sin\theta)(1 + \cos\theta) = \frac{1}{2}(1 + \cos\theta)(\sin\theta)$$

$$\frac{4}{\pi^2}(\theta + \sin\theta) = \sin\theta \qquad [\cos\theta \neq -1]$$

$$\frac{4}{\pi^2}\theta + \sin\theta\left(\frac{4}{\pi^2} - 1\right) = 0. \qquad \theta = 0 \text{ is a solution.}$$

$$\frac{\sin\theta}{\theta} = \frac{4}{\pi^2 - 4} = \frac{4}{9.8696 - 4} = \frac{4}{5.8696}$$

$$\frac{\sin\theta}{\theta} = .6815$$

$\theta = \pm 1.46$ (Jahnke and Emde, appendix p. 33)

$$F_0 = 1; \qquad F_{\pm 1.46} = \left(\frac{1.46 + .99}{\pi}\right)^2 + \left(\frac{1 + .11}{2}\right)^2 = .92.$$

Taking the second derivative:

$$G = \frac{2}{\pi^2}\left[(1 + \cos\theta)^2 + (\theta + \sin\theta)(-\sin\theta)\right] - \frac{1}{2}\left[-\sin^2\theta + (1 + \cos\theta)\cos\theta\right].$$

For $\theta = 0$, $G < 0$, so $\theta = 0$ is a maximum.
For $\theta = \pm 1.46$, $G > 0$, and $\theta = \pm 1.46$ are minimums.
Thus $F \leqq 1$.
(Note: The "$=$" sign is necessary.)

Also solved by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; C. B. A. Peck, State College, Pennsylvania; Simeon Reich, Haifa, Israel; Sidney Spital, California State Polytechnic College; K. L. Yocom, South Dakota State University; and the proposer.

Raymond E. Whitney, Lock Haven State College, Pennsylvania, pointed out the necessity of including the equals sign along with the inequality.

Comment on Proposal 587:
Mathematics Magazine, 39, (1966), 188.

Comment on Problem 587

587. [May, 1965, and January, 1966] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Prove the following inequality

$$\left(\frac{\theta + \sin\theta}{\pi}\right)^2 + \cos^4\frac{1}{2}\theta < 1, \qquad (-\pi < \theta < +\pi).$$

Comment by the proposer.
The given inequality is equivalent to

$$\left(\frac{\theta + \sin\theta}{\pi}\right)^2 + \left(\frac{1 + \cos\theta}{2}\right)^2 < 1.$$

Now consider the cycloid

$$x = \theta + \sin\theta$$
$$y = 1 + \cos\theta$$

and the ellipse

$$\frac{x^2}{\pi} + \frac{y^2}{4} = 1.$$

They have common origin and equal diameters. The two curves have points in common at the three vertices. We can prove that at the neighborhoods of these points the cycloid lies inside the ellipse. Since their concavity is in the same direction, the cycloid lies wholly inside the ellipse except at the three points. The above inequality is the analytical interpretation for the property just mentioned.

Solution to Proposal 599:
Mathematics Magazine, 39, (1966), 134.

Linearly Dependent Vectors

599. [September, 1965] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

If a, b, and c are any three vectors in 3-space, then show that the vectors

$$a\times(b\times c),\ b\times(c\times a),\ c\times(a\times b)$$

are linearly dependent.

Solution by Carl G. Wagner, Duke University.

By a well-known theorem of the vector calculus (see page 90 of Nickerson, Steenrod, and Spencer's *Advanced Calculus* for a proof based on axioms for a vector product):

$$A \times (B \times C) = (A\cdot C)B - (A\cdot B)C.$$

Writing out the other vector products,

$$B \times (C \times A) = (B\cdot A)C - (B\cdot C)A = (A\cdot B)C - (B\cdot C)A$$
$$C \times (A \times B) = (C\cdot B)A - (C\cdot A)B = (B\cdot C)A - (A\cdot C)B.$$

Hence,

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0$$

(This is known as the Jacobi Identity.)

Also solved by Joseph B. Bohac, St. Louis, Missouri; Dermott A. Breault, Sylvania Applied Research Laboratory, Waltham, Massachusetts; Dewey C. Duncan, Los Angeles, California; Philip Fung, Cleveland State University, Ohio; Mrs. A. C. Garstang, Boulder, Colorado; Carl Harris, Western Electric Company, Princeton, New Jersey; Stephen Hoffman, Trinity College, Connecticut; John E. Homer, Jr., St. Procopius College, Illinois; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Mich.; John Kieffer, University of Missouri at Rolla; E. S. Langford, U. S. Naval Postgraduate School; Lieselotte Miller, Georgia Institute of Technology; Stanley Rabinowitz, Far Rockaway, New York; Kenneth A. Ribet, Brown University; Richard Riggs, Jersey City State College; Howard L. Walton, Yorktown High School, Arlington, Virginia; K. L. Yocum, South Dakota State University; and the proposer.

Solution to Proposal 600:
***Mathematics Magazine*, 39, (1966), 189.**

Related Triangles

600. [November, 1965] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

If the area of a triangle ABC is S and the areas of the in- and ex-contact triangles are T, T_a, T_b, T_c, then show that

$$T_a + T_b + T_c - T = 2S \tag{1}$$

$$T_a^{-1} + T_b^{-1} + T_c^{-1} - T^{-1} = 0. \tag{2}$$

Solution by the proposer.

Let I be the incenter and DEF be the in-contact triangle of ABC and let R, r be circumradius and inradius respectively. Then

$$\begin{aligned} IEF/S &= \tfrac{1}{2}r^2 \sin(\pi - A)/(\tfrac{1}{2}bc \sin A) \\ &= r^2/bc = ar^2/abc = ar^2/4RS \end{aligned}$$

or

$$IEF = ar^2/4R$$

and similarly

$$\begin{aligned} IFD &= br^2/4R \\ IDE &= cr^2/4R. \end{aligned}$$

Thus

$$T = IEF + IFD + IDE = (a+b+c)r^2/4R = 2ur\cdot r/4R = S/2R$$

and similarly

$$T_a = Sr_a/2R, \qquad T_b = Sr_b/2R, \qquad T_c = Sr_c/2R.$$

We then have

$$\begin{aligned} T_a + T_b + T_c - T &= S(r_a + r_b + r_c)/2R - Sr/2R \\ &= S(4R + r - r)/2R = 2S \end{aligned} \tag{1}$$

$$T_a^{-1} + T_b^{-1} + T_c^{-1} - T^{-1} = 2R(1/r_a + 1/r_b + 1/r_c - 1/r)/S = 0. \tag{2}$$

Also solved by P. N. Bajaj, Western Reserve University; Stanley Rabinowitz, Far Rockaway, New York; G. L. N. Rao, J. C. College, Jamshedpur, India.

Solution to Proposal 609:
Mathematics Magazine, 39, (1966), 248.

CRYPTA-EQUIVALENCE

609. [January, 1966] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the following cryptarithm in the decimal system:

$$4\cdot NINE = 9\cdot FOUR$$

Solution by J. D. E. Konhauser, University of Minnesota.

The products $4\cdot E$ and $9\cdot R$ must have the same units digit. Therefore, the only possible (E, R) combinations are (1, 6), (3, 8), (5, 0), (7, 2), and (9, 4). Since 4 and 9 are relatively prime, 9 must divide $2N+I+E$.

Case (1, 6): If $E=1$, 9 must divide $2N+I+1$, leading to the following (N, I) combinations: (4, 0), (3, 2), (7, 3), (2, 4), (5, 7), (9, 8), and (4, 9). The corresponding values for $FOUR$ are 1796, 1436, 3276, 1076, 2556, 4396, and 2196. The first two, the fourth, and the last must be rejected since $E=1$. The third is out since $N=7$. The fifth is out since 5 is repeated. The sixth is out since $N=9$.

Similar analysis applied to the remaining cases leads to the solutions given below:

Case (3, 8): $4\cdot 4743=9\cdot 2108$.
Case (5, 0): $4\cdot 6165=9\cdot 2740$.
Case (7, 2): $4\cdot 6867=9\cdot 3052$.
Case (9, 4): $4\cdot 5859=9\cdot 2604$.

Also solved by Monte Dernham, San Francisco, California; Samuel P. Hoyle, Jr., University of Virginia; Sidney Kravitz, Dover, New Jersey; C. C. Oursler, Southern Illinois University (Edwardsville); Richard Riggs, Jersey City State College, New Jersey; Jerome J. Schneider, Chicago, Illinois; and Charles W. Trigg, San Diego, California.

Partial solutions were submitted by Merrill Barneby, Wisconsin State University (La Crosse); Charles R. Berndtson, Massachusetts Institute of Technology; Lindley J. Burton, Lake Forest College, Illinois; Anton Glaser, Pennsylvania State University (Ogontz); J. A. H. Hunter, Toronto, Canada; Beatriz Margolis, University of Maryland; John W. Milsom, Slippery Rock State College, Pennsylvania; William L. Mrozek, Wyandotte, Michigan; Sam Newman, Atlantic City, New Jersey; C. R. J. Singleton, Petersham, Surrey, England; Lowell Van Tassel, San Diego City College; Gary B. Weiss, New York University, School of Medicine; Donald R. Wilder, Rochester, New York; Dale Woods, Missouri State Teachers College; and the proposer.

Solution to Proposal 628:
Mathematics Magazine, 40, (1967), 102.

Pythagorean Alphametic

628. [September, 1966] *Proposed by B. Suer and Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the alphametic,

$$COS^2 + SIN^2 = UNO^2$$

in the decimal system.

Solution by J. A. H. Hunter, Toronto, Ontario, Canada.

We have $S^2+N^2 \equiv O^2$ (mod 10), and obviously $S \neq$ zero. For each N, for $N=0, 1, \cdots, 9$, we tabulate possible S and corresponding O values, bearing in mind digital "square-endings."

Since $U>S$, we then test each possible UNO value to find its representations (if any) as sum of two squares: bearing in mind the conditions which are required for this to be possible. Where representation as sum of squares is possible, we can then note corresponding SIN and COS from the well-known solution:

$$(x^2 + y^2)^2k^2 = (x^2 - y^2)^2k^2 + (2xy)^2k^2.$$

The working is somewhat tedious, but not unduly so. It is found that uniquely we have

$$391^2 + 120^2 = 409^2.$$

Also solved by R. H. Anglin, Danville, Virginia; Merrill Barnebey, Wisconsin State University at LaCrosse; Sarah Brooks, Utica Free Academy, New York; Jack Dix, Rutgers University; Charles R. Fleenor, Ball State University, Indiana; Michael Goldberg, Washington, D. C.; Jerome J. Schneider, Chicago, Illinois; Charles W. Trigg, San Diego, California; and the proposers.

Solution to Proposal 639:
Mathematics Magazine, 40, (1967), 166.

A Convex Quadrilateral Inequality

639. [November, 1966] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let $ABCD$ be a convex quadrangle and P be the intersection of diagonals AC and BD. Let the distance of P from the sides $AB=a$, $BC=b$, $CD=c$, $DA=d$ be x, y, z, and t respectively. Prove that

$$x+y+z+t<\tfrac{3}{4}(a+b+c+d).$$

Solution by Leon Bankoff, Los Angeles, California.

Let the bisectors of the angles between the diagonals AC and BD meet AB, BC, CD, DA in R, S, T, U.

By a corollary of the Erdös-Mordell Theorem,

$$\begin{aligned}
2(PS+PT) &< PB+PC+PD \\
2(PT+PU) &< PC+PD+PA \\
2(PU+PR) &< PD+PA+PB \\
2(PR+PS) &< PA+PB+PC
\end{aligned}$$

or $4(PR+PS+PT+PU)<3(PA+PB+PC+PD)$.

This inequality is stronger than the one proposed because

$$x+y+z+t\leqq PR+PS+PT+PU$$

and $PA+PB+PC+PD<a+b+c+d$.

Also solved by Leon Bankoff, Los Angeles, California (second solution); Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania; Stanley Rubinowitz, Far Rockaway, New York and the proposer.

Solution to Proposal 649:
Mathematics Magazine, 40, (1967), 279.

An Alphametic

649. [March, 1967] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the cryptarithm

$$\begin{array}{r} THREE \\ +\quad FOUR \\ \hline SEVEN \end{array}$$

in the decimal system such that:

3 does not divide $THREE$ in which the digit 3 is missing;
4 does not divide $FOUR$ in which the digit 4 is missing;
7 does not divide $SEVEN$ in which the digit 7 is missing.

Solution by Harry M. Gehman, SUNY at Buffalo, New York.

Let us first solve the cryptarithm, given only that
(a) the digit 3 is missing from $T\ H\ R\ E\ E$;
(b) the digit 4 is missing from $F\ O\ U\ R$;
(c) the digit 7 is missing from $S\ E\ V\ E\ N$.
The problem has seven solutions:

$$
(1)\quad \begin{array}{r} 16544 \\ 7805 \\ \hline 24349 \end{array}
\qquad\qquad
(2)\quad \begin{array}{r} 47266 \\ 9102 \\ \hline 56368 \end{array}
$$

$$
(3)\quad \begin{array}{r} 75244 \\ 9102 \\ \hline 84346 \end{array}
\qquad\qquad
(4)\quad \begin{array}{r} 79244 \\ 5102 \\ \hline 84346 \end{array}
$$

$$
(5)\quad \begin{array}{r} 17544 \\ 6805 \\ \hline 24349 \end{array}
\qquad\qquad
(6)\quad \begin{array}{r} 49266 \\ 7102 \\ \hline 56368 \end{array}
$$

$$
(7)\quad \begin{array}{r} 24811 \\ 6708 \\ \hline 31519 \end{array}
$$

The condition (d) that 3 does not divide $T\ H\ R\ E\ E$ eliminates solutions (5) and (6). The condition (e) that 4 does not divide $F\ O\ U\ R$ eliminates (7). The condition (f) that 7 does not divide $S\ E\ V\ E\ N$ does not eliminate any solution.

Therefore the problem as proposed has four solutions: (1)–(4).

If we ignore conditions (d) (e) (f) but retain conditions (a) (b) (c) with the additional condition indicated we have unique solutions as follows:

(g) $T\ H\ R\ E\ E$ contains the digit 8. Solution (7).
(h) $S\ E\ V\ E\ N$ contains the digit 1. Solution (7).
(i) $F\ O\ U\ R$ contains both the digits 5 and 6. Solution (5).
(j) $T\ H\ R\ E\ E$ contains neither 6 nor 7. Solution (7).
(k) $T\ H\ R\ E\ E$ contains both 6 and 7. Solution (2).
(l) $T\ H\ R\ E\ E$ contains both 1 and 2. Solution (7).
(m) $T\ H\ R\ E\ E$ contains neither 5, 6 nor 7. Solution (7).
(n) $T\ H\ R\ E\ E$ contains neither 5, 7 nor 9. Solution (7).

and so on.

The fact that solution (7) occurs so frequently in this list seems to indicate that it has a pattern of digits essentially different from the other six solutions. From the standpoint of numerology, this has some deep significance, I am sure.

Solution to Proposal 680:
Mathematics Magazine, **41, (1968), 219.**

A Circular Locus

680. [January, 1968] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let E be an ellipse and t', t'' be two variable parallel tangents to it. Consider a circle C, tangent to t', t'' and to E externally. Show that the locus of the center of C is a circle.

Solution by the proposer.

Let the ellipse be given by the equation

(1) $$x^2/a^2 + y^2/b^2 = 1$$

Denoting the center and radius of (C) by (α, β) and r, from $r = (0, t')$, we have

(2) $$r^2 = (a^2\beta^2 + b^2\alpha^2)/(\alpha^2 + \beta^2);$$

r is also given by

(3) $$(x - \alpha)^2 + (y - \beta)^2 = r^2$$

such that the normal at $T(x, y)$ of (E) passes through the center C.

CT is an extremal distance of $C(\alpha, \beta)$ to (E). To determine it we use the method of Lagrange multipliers. Let

$$F(x, y) = (x - \alpha)^2 + (y - \beta)^2 + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

where λ is determined by

(4) $$\begin{aligned} 1/2F_x &= x - \alpha + \lambda x/a^2 = 0, \\ 1/2F_y &= y - \beta + \lambda y/b^2 = 0 \end{aligned}$$

and (1). Eliminating x, y, α, β we obtain a quartic equation in λ. So we proceed in a different way. Supposing that the statement is true, we have

(5) $$OC^2 = \alpha^2 + \beta^2 = (a + b)^2$$

Solving α, β from (4) and setting in (5) and comparing the result with (1) we get $\lambda = ab$.

We observe that if we choose $\lambda = ab$, the three equations (1), (2), (3) are consistent and this consistency gives $\alpha^2 + \beta^2 = (a+b)^2$ proving that the locus of $C(\alpha, \beta)$ is a circle.

Also solved by Michael James Smithson, Bellevue, Washington.

Comment on Proposal 680:
Mathematics Magazine, 42, (1969), 162.

Comment on Problem 680

680. [January and September, 1968] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let E be an ellipse and t', t'' be two variable parallel tangents to it. Consider a circle C, tangent to t', t'' and to E externally. Show that the locus of the center of C is a circle.

Comment by A. W. Walker, Toronto, Canada.

Many interesting properties of an ellipse are associated with its so-called Chasles circles of radius $a \pm b$ concentric with the ellipse. If the center of the variable circle C lies on the *inward* normal, its locus is the inner Chasles circle. The result in Problem 680 was established by Mannheim, Nouvelles Annales de Math., 4, 3, (1903) 483, and is equivalent to the following old Japanese theorem (Iwata, 1862):

If an ellipse touches externally two equal nonoverlapping fixed circles and their parallel common tangents, the sum of its major and minor axes is equal to the distance between the centers of the circles.

See Tôhoku Math. J., (1), 11, (1917) 65, where with rather obscure justification, it is asserted that the theorem is untrue!

Solution to Proposal 724:
Mathematics Magazine, 42, (1969), 274.

Triangle Probability

724. [March, 1969] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Find the probability that for a point P taken at random in the interior of a triangle ABC ($a \geqq b \geqq c$), the distances of P from the sides of ABC form the lengths of sides of a triangle.

Solution by L. Carlitz, Duke University.

Let the internal angle bisectors meet the sides BC, CA, AB in L, M, N, respectively. Let x, y, z denote the distances from the point P to the sides BC, CA, AB. The equation $x=y+z$ represents a straight line, namely MN. Similarly $y=z+x$ represents NL, $z=x+y$ represents LM. The incenter I is in the interior of the triangle LMN; hence by continuity the point P must be restricted to the interior of LMN, so that the desired probability is equal to

$$p = \frac{\text{area } LMN}{\text{area } ABC} = \frac{\Delta_0}{\Delta}\,.$$

Now

$$CL = \frac{ab}{b+c}, \qquad CM = \frac{ab}{a+c},$$

so that

$$\text{area } CLM = \frac{1}{2}\,\frac{a^2b^2 \sin\gamma}{(a+c)(b+c)} = \frac{ab\Delta}{(a+c)(b+c)}.$$

By symmetry

$$\text{area } AMN = \frac{bc\Delta}{(b+a)(c+a)}, \qquad \text{area } BLN = \frac{ac\Delta}{(a+b)(c+b)}.$$

Thus

$$\begin{aligned}\Delta_0 = \text{area } LMN &= \Delta - \sum \frac{bc\Delta}{(b+a)(c+a)}\\ &= \Delta\left\{1 - \frac{\sum bc(b+c)}{(a+b)(b+c)(c+a)}\right\}\\ &= \Delta\,\frac{2abc}{(a+b)(b+c)(c+a)}\end{aligned}$$

so that

$$p = \frac{2abc}{(a+b)(b+c)(c+a)}.$$

Since $a+b \geqq 2\sqrt{ab}$, it follows that

$$p \leqq \tfrac{1}{4}$$

with equality only when $a=b=c$.

Also solved by Michael Goldberg, Washington, D.C.; C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania; F. G. Schmitt, Jr., Berkeley, California (partially); Paul J. Zwier, Palo Alto, California; and the proposer.

Solution to Proposal 738:
***Mathematics Magazine*, 43, (1970), 109.**

738. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

There is a river with parallel and straight shores. A is located on one shore and B on the other, with $AB=72$ miles. A ferry boat travels the straight path AB from A to B in four hours and from B to A in nine hours. If the boat's speed on still water is $v=13$ mph, what is the velocity of the flow?

$$13 \text{ mph} + s \geqq 18 \text{ mph}, \quad \text{and thus}$$
$$s \geqq 5 \text{ mph}$$

Returning from B to A, the maximum possible magnitude for the vector sum is 13 mph$-s$. Thus,

$$13 \text{ mph} - s \geqq 8 \text{ mph}$$
$$s \leqq 5 \text{ mph}.$$

Since $s \geqq 5$ mph and $s \leqq 5$ mph, $s = 5$ mph.

An alternate method would be to allow the width of the river to be m miles and find s for any m. It is possible to find m using this method. It involves solving the following equation:

$$13\sqrt{5184 - m^2} = 4\sqrt{13689 - m^2} + 9\sqrt{2704 - m^2}.$$

The only solution is $m = 0$ mph. With this value of m it follows that $s = 5$ mph.

Also solved by Richard L. Breisch, University of Colorado; Bruce A. Broemser, El Sobrante, California; Raphael T. Coffman, Richland, Washington; George F. Corliss, Michigan State University; Mickey Dargitz, Ferris State College, Michigan; Gerald C. Dodds, HRB-Singer, Inc., State College, Pennsylvania; Frank Eccles, Phillips Academy, Massachusetts; W. W. Funkenbusch, Michigan Technological University; Michael Goldberg, Washington, D. C.; John M. Howell, Littlerock, California; Alfred Kohler, Long Island University, New York; Lew Kowarski, Morgan State College, Maryland; J. R. Kuttler, Johns Hopkins University; Joseph O'Rourke, St. Joseph's College, Pennsylvania; C. D. O'Shaughnessy, University of Saskatchewan; John E. Prussing, University of Illinois; John R. Ray, Clemson University; Simeon Reich, Israel Institute of Technology, Haifa, Israel; Ray B. Robinson, Butler, Tennessee; Steve Rohde, Lehigh University; E. F. Schmeichel, College of Wooster, Ohio; W. A. Schmidt, Texas A and M University; E. P. Starke, Plainfield, New Jersey; Charles W. Trigg, San Diego, California; A. W. Walker, Toronto, Canada; Sam Zaslavsky, City University of New York; and the proposer.

Solution to Proposal 743:
***Mathematics Magazine*, 43, (1970), 169.**

Tetrahedral Inequality

743. [November, 1969] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let P be an interior point of a regular tetrahedron, $T \equiv A_1A_2A_3A_4$, with $p_i = PA_i$, and let x_{ij} denote the distance of P from the edge A_iA_j. Then prove

$$\sum_{i=1}^{4} p_i \geqq 2\sqrt{3}/3 \sum_{i<j} x_{ij},$$

equality holding if and only if P is at the center O of T.

Solution by Michael Goldberg, Washington, D.C.

Given a base of fixed length and the sum of two other lengths, the triangle of greatest height is obtained when the triangle is isosceles. Similarly, given the same base, and the sum of three other lengths to form three triangles by using the three pairs of these three sides, the sum of the heights is maximized when the triangles are congruent isosceles triangles. This can be generalized to n triangles. Hence, the sum of the distances of P from the edges of a regular tetrahedron divided by the sum of the distances of P from the vertices is maximized when P is at the center of the tetrahedron.

If e is the length of the edge of the tetrahedron then the distance h between opposite edges is given by

$$h^2 + (e/2)^2 + (e/2)^2 = e^2, \quad \text{or} \quad h = e/\sqrt{2}.$$

The distance R from the center to a vertex is given by

$$R^2 = (e/2)^2 + (h/2)^2 = e^2/4 + e^2/8, \quad \text{or} \quad R = \sqrt{3}e/2\sqrt{2}.$$

Hence, when P is at the center, the ratio of the sums is

$$4R/3h = (2e\sqrt{3}/\sqrt{2})/(3e/\sqrt{2}) = 2\sqrt{3}/3.$$

A similar extremal is obtained for each of the regular polyhedra, and for each of the regular polygons. Of course, the ratio of the two sums depends upon the figure considered.

Also solved by the proposer. One incorrect solution was received.

Comment on Proposal 743:
Mathematics Magazine, 44, (1971), 44.

Comment on Problem 743

743. [November, 1969, and May, 1970] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let P be an interior point of a regular tetrahedron, $T \equiv A_1A_2A_3A_4$, with $p_i = PA_i$, and let x_{ij} denote the distance of P from the edge A_iA_j. Then prove

$$\sum_{i=1}^{4} p_i \geqq 2\sqrt{3}/3 \sum_{i<j} x_{ij},$$

equality holding if and only if P is at the center O of T.

Comment by E. F. Schmeichel, College of Wooster, Ohio.

The inequality should read

$$\sum_{i=1}^{4} p_i \geqq \frac{2\sqrt{2}}{3} \sum_{i<j} x_{ij}.$$

Apparently a $\sqrt{3}$ was misprinted in place of the $\sqrt{2}$ above. To show that the inequality as printed is false consider a regular tetrahedron of edge length 1. Let P be the midpoint of edge A_1A_2. Then $p_1 = p_2 = 1/2$, $p_3 = p_4 = \sqrt{3}/2$, $x_{12} = 0$, $x_{13} = x_{14} = x_{23} = x_{24} = \sqrt{3}/4$ and $x_{34} = \sqrt{2}/2$.

Thus

$$\sum_i p_i = 1 + \sqrt{3} < 2.8 \quad \text{and} \quad \sum_{i<j} x_{ij} = \sqrt{3} + \sqrt{2}/2$$

Since $\sqrt{6} > 2.4$ we have

$$\frac{2\sqrt{3}}{3} \sum_{i<j} x_{ij} = 2 + \sqrt{6}/3 > 2.8$$

so for the point in question

$$\sum_i p_i < \frac{2\sqrt{3}}{3} \sum_{i<j} x_{ij}.$$

By the continuity of the distances involved, this inequality will be retained for interior points of the tetrahedron sufficiently near P.

Solution to Proposal 756:
***Mathematics Magazine*, 43, (1970), 283.**

Centrally Symmetric Curves

756. [March, 1970] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Determine closed and centrally symmetric curves C, other than circles, such that the product of two perpendicular radius vectors (issued from the center) be a constant.

Solution by Harry W. Hickey, Arlington, Virginia.

Let us consider a generalized form of the problem: "Determine closed and centrally symmetric curves C, other than circles, such that the product of two radius vectors (issued from the center) be a constant, when the angle between the radius vectors is π/N, N being any positive even integer." We will call the center O, while the constant product is R^2. Construct a circle K of center O and radius R. Now for every point of C inside the circle, there is a point outside it, such that the product of the distances of the two points from O is R^2. Hence C crosses K at some point, say A, and crosses again at point B, where $\angle AOB = \pi/N$. Draw an arc from A to B which does not pass through O, nor intersect any line through O more than once—aside from these restrictions, the form of the arc is arbitrary. Let the polar equation of this arc be $\rho = f(\theta)$. The restrictions we have placed on the form of the arc insure that the reciprocal of f is always finite, and that f is single-valued—a multivalued f leads to ambiguities about the length of the radius vector. So far, f is defined for values of θ in a domain of length π/N. We can extend this to other values of θ by writing

$$f(\theta + \pi/N) = R^2/f(\theta) \quad \text{for all } \theta,$$

and the the curve C is defined! Because f is now periodic, of period $2\pi/N$, and since N is even, we have $f(\theta+\pi) = f(\theta)$, making C centrally symmetric (drop the symmetry requirement, and N can be odd).

Also solved by Michael Goldberg, Washington, D. C.; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Lew Kowarski, Morgan State College, Maryland; John Oman, Wisconsin State University, Oshkosh; E. F. Schmeichel, College of Wooster, Ohio; and the proposer.

Solution to Proposal 763:
Mathematics Magazine, 44, (1971), 108.

Quasi Zeta Functions

763. [May, 1970] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Prove:

$$\left(1+\frac{1}{3^{10}}+\frac{1}{5^{10}}+\cdots\right)=\left(1+\frac{1}{3^4}+\frac{1}{5^4}\cdots\right)\left(1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\cdots\right).$$

Solution by M. G. Greening, University of New South Wales, Australia.

$\zeta(s)=\sum_{s=1}^{\infty} n^{-s}$, convergent for $|s|>1$.

$$1+3^{-4}+5^{-4}+\cdots=\zeta(4)-\frac{1}{2^4}\zeta(4)=15\zeta(4)/2^4$$

$$1-2^{-6}+3^{-6}-4^{-6}+\cdots=\zeta(6)-2\left(\frac{1}{2^6}\zeta(6)\right)=(1-2^{-5})\zeta(6)$$

$$1+3^{-10}+5^{-10}+\cdots=\zeta(10)-\frac{1}{2^{10}}\zeta(10)=(1-2^{-5})\frac{33}{22}\zeta(10).$$

As $\zeta(2n)=2^{2n-1}B_n\pi^{2n}/(2n)!$ where B_n is the nth Bernoulli number, and $B_2=1/30$, $B_3=1/42$, $B_5=5/66$, the result follows after simplification.

Also solved by Bernard August, Glassboro State College, New Jersey; Miguel Bamberger, Monterey, California; Walter Blumberg, New Hyde Park, New York; Wray G. Brady, Slippery Rock State College, Pennsylvania; Richard L. Breisch, Pennsylvania State University; Donald R. Childs, Naval Underwater Weapons Research and Engineering Station, Rhode Island; Gerald C. Dodds, HRB-Singer, Inc., State College, Pennsylvania; D. Dummit, San Mateo High School, California; Louise Grinstein, New York, New York; Jeffrey Hoffstein, Bronx High School of Science, New York; Robert F. Jackson, University of Toledo, Ohio; Shiv Kumar, Panjabi University, and Miss Nirmal, Government Girls' High School, Panipat, India (jointly); J. R. Kuttler, Johns Hopkins Applied Physics Laboratory, Maryland; Herbert R. Leifer, Pittsburgh, Pennsylvania; Michael J. Martino, IBM, Poughkeepsie, New York; Kenneth Rosen, University of Michigan; L. E. Schaefer, General Motors Institute, Flint, Michigan; E. F. Schmeichel, College of Wooster, Ohio; E. P. Starke, Plainfield New Jersey; Paul D. Thomas, Naval Research Laboratory, Washington, D.C.; Graham C. Thompson, Binghamton, New York; Michael R. Wise, University of Colorado; Gregory Wulczyn, Bucknell University; K. L. Yocom, University of Wyoming; and the proposer.

Solution to Proposal 775:
Mathematics Magazine, 44, (1971), 230.

Inverse Functions

775. [November, 1970] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Prove $\int_0^1 \sqrt[q]{1-x^p}\,dx = \int_0^1 \sqrt[p]{1-x^q}\,dx$, where $p, q > 0$.

I. *Solution by J. C. Binz, Bern, Switzerland.*

Let more generally f be a decreasing continuous function in $[a, b]$. Then the inverse function g exists in $[f(b), f(a)]$ and is also decreasing and continuous.

Compute

$$\int_{f(b)}^{f(a)} g(y)dy = \int_b^a g[f(t)]f'(t)dt = \int_b^a tf'(t)dt = af(a) - bf(b) + \int_a^b f(t)dt.$$

Hence, if additionally we have $f(a)=b$, $f(b)=a$, then $\int_a^b g(t)dt = \int_a^b f(t)dt.$

The functions $f: x \to \sqrt[p]{1-x^q}$ and $g: x \to \sqrt[q]{1-x^p}$ represent in $[0, 1]$ a special case of the preceding situation, which proves the proposition.

II. *Solution by Vàclav Konečný, Jarvis Christian College, Hawkins, Texas.*

$$\begin{aligned}\int_0^1 \sqrt[q]{1-x^p}\,dx &= \frac{1}{p}\int_0^1 z^{-1+1/p}(1-z)^{1/q}dz \\ &= \frac{1}{p}B(1/p, 1+1/q) = \frac{1}{pq}B(1/p, 1/q)\end{aligned}$$

where $p, q > 0$ to get the real value. We used the substitution $x^p = z$. B is the beta function and as $B(x, y) = B(y, x)$ the value of the integral is unchanged if we interchange p and q.

Also solved by Joseph Beer and Bernard August (jointly), Glassboro State College, New Jersey; Walter Blumberg, Flushing High School, New York; Dermott A. Breault, Cyber, Inc., Cambridge, Massachusetts; Robert X. Brennan, Dover, New Jersey; Robert J. Bridgman, Mansfield State College, Pennsylvania; David C. Brooks, Seattle Pacific College, Washington; G. R. Desai, St. Louis University; Robert Desko, Davenport, Iowa; Ellis Detwiler, Adams, New York; Santo M. Diano, Havertown, Pennsylvania; Fred Dodd, University of South Alabama; M. G. Greening, University of New South Wales, Australia; Robert G. Griswold, University of Hawaii, Hilo College; Philip Haverstick, Fort Belvoir, Virginia; Harry W. Hickey, Arlington, Virginia; John E. Homer, Lisle, Illinois; N. J. Kuenzi, Oshkosh, Wisconsin; David E. Mannes, SUNY, Oneonta, New York; Stephen B. Maurer, Phillips Exeter Academy; Edward Moylan, Ford Motor Company, Dearborn, Michigan; Albert J. Patsche, Rock Island Arsenal, Illinois; V. V. Ramana Rao, Andhra University, South India; B. E. Rhoades, Indiana University; Steve M. Rohde, General Motors Research Laboratories, Warren, Michigan; E. F. Schmeichel, College of Wooster, Ohio; Harry Siller, Hofstra University; A. Swyanavayanamuti, Andhra University, Waltair, South India; R. A. Struble, North Carolina State University; Philip Tracy, APO San Francisco; C. S. Venkataraman, Sree Kerala Varma College, Trichur, South India; John R. Ventura, Jr., Naval Underwater Systems Center, Newport, Rhode Island; R. L. Woodriff, Menlo College, Menlo Park, California; Thomas Wray, Department of Energy, Mines and Resources, Ottawa, Canada; Robert L. Young, Cape Cod Community College, Massachusetts; Paul Zwier, Calvin College, Michigan; and the proposer.

Comment on Proposal 775:
***Mathematics Magazine*, 45, (1972), 293.**

Comment on Problem 775

775. [November, 1970, and September, 1971] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Prove $\int_0^1 \sqrt[q]{1-x^p}\,dx = \int_0^1 \sqrt[p]{1-x^q}\,dx$, where $p,q>0$.

Comment by Ralph Leung, Berkeley, California.

The problem would become immediate if we rewrite the above equality as

$$\int_0^1 \sqrt[q]{1-x^p}\,dx = \int_0^1 \sqrt[p]{1-y^q}\,dy.$$

Both sides give the area of the region bounded by the x-axis, the y-axis, and the graph of $x^p + y^q = 1$ in the first quadrant — the l.h.s. by integrating with respect to x, the r.h.s. with respect to y.

Solution to Proposal 806:
***Mathematics Magazine*, 45, (1972), 171.**

Symmetry About a Line

806. [September, 1971] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let H be the orthocenter of an isosceles triangle ABC, and let AH, BH, and CH intersect the opposite sides in D, E, and F, respectively. Prove that the incenters of the right triangles HBD, HDC, HCE, HEA, HAF, and HFB lie on a conic.

Solution by Vladimir F. Ivanoff, San Carlos, California.

The problem is a special case of the following theorem:

If six points are symmetric about a line, they lie on a conic.

It can be easily proved by the converse of Pascal's theorem, or else by analytical method.

Incidentally, the theorem holds true, if six points are symmetric about a point. By choosing the point of symmetry as the origin, the six points have coordinates as follows:

$$(x_1, y_1), \quad (x_2, y_2), \quad (x_3, y_3),$$
$$(-x_1, -y_1), (-x_2, -y_2), (-x_3, -y_3,),$$

and the equation of the conic is

$$\begin{vmatrix} x^2 & y^2 & xy & 1 \\ x_1^2 & y_1^2 & x_1y_1 & 1 \\ x_2^2 & y_2^2 & x_2y_2 & 1 \\ x_3^2 & y_3^2 & x_3y_3 & 1 \end{vmatrix} = 0.$$

Also solved by Leon Bankoff, Los Angeles, California (three solutions); Ragnar Dybvik, Tingvoll, Norway; Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; and the proposer.

Solution to Proposal 839:
Mathematics Magazine, 46, (1973), 169.

Probability of No Change

839. [September, 1972] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Given three boxes each containing w white balls and r red balls identical in shape. Take a ball from the first box and put it in the second box, then take a ball from the second box and put it in the third, and finally take a ball from the third box and put it in the first. Find the probability that the boxes have their original contents as to color.

Solution by Thomas Spencer, Trenton State College, New Jersey.

A moment's reflection will show that the only events which will leave the color composition of all three boxes unchanged are the choices white, white, white or red, red, red. Their probabilities by the multiplication rule are:

$$\left(\frac{w}{w+r}\right)\left(\frac{w+1}{w+r+1}\right)\left(\frac{w+1}{w+r+1}\right) = \frac{w(w+1)^2}{(w+r)(w+r+1)^2}$$

and

$$\left(\frac{r}{w+r}\right)\left(\frac{r+1}{w+r+1}\right)\left(\frac{r+1}{w+r+1}\right) = \frac{r(r+1)^2}{(w+r)(w+r+1)^2}$$

respectively. These events are disjoint and thus the required probability is their sum:

$$\frac{w(w+1)^2 + r(r+1)^2}{(w+r)(w+r+1)^2}.$$

Note: In the case of N boxes the solution would be:

$$\frac{w(w+1)^{N-1} + r(r+1)^{N-1}}{(w+r)(w+r+1)^{N-1}}.$$

Also solved by Gladwin Bartel, La Junta, Colorado; Melvin Billick, Midland High School, Michigan; J. L. Brown, Jr., Pennsylvania State University; Joseph B. Browne, Oklahoma State University; Daniel L. Calloway, Ashville, North Carolina; Abraham L. Epstein, Hanscom Field, Massachusetts; George Fabian, Park Forest, Illinois; Michael Goldberg, Washington, D. C.; Kathleen Harris, New Hampton, Iowa; Karl Heuer, Moorhead, Minnesota; John M. Howell, Littlerock, California; Vaclav Konecny, Jarvis Christian College, Texas; Lew Kowarski, Morgan State College, Maryland; Michael W. O'Donnell, Carnegie-Mellon University; George Pfeiffer, Old Dominion University, Virginia; Louisa Russo, Michigan Technological University; R. Shantaram, University of Michigan-Flint; and the proposer.

Solution to Proposal 859:
***Mathematics Magazine*, 47, (1974), 49.**

A Non-Unique Cryptarithm

859. [March, 1973] *Proposed by B. Suer and H. Demir, Middle East Technical University, Ankara, Turkey.*

Solve the cryptarithm $THREE + NINE = EIGHT + FOUR$.

I. *Solution by Harry L. Nelson, Livermore, California.*

There are 12 solutions in decimal base. They are:

$THREE$	+	$NINE$	=	$EIGHT$	+	$FOUR$
30122	+	4542	=	25703	+	8961
29433	+	7073	=	30692	+	5814
40233	+	5653	=	36104	+	9782
59766	+	4346	=	63295	+	0817
70566	+	2926	=	69107	+	4385
69877	+	5457	=	74096	+	1238

In each of these one can interchange the values of G and O to obtain another solution yielding 12 in all.

If one were to add the condition that "*THREE* is a prime" only the pair 69877 + 5457 = 74096 + 1238 = 74296 + 1038 would qualify; and if in addition we ask that "FOUR not be divisible by 3" the solution would be unique (base 10).

II. *Solution by John Tabor and John Beidler (jointly), University of Scranton, Pennsylvania.*

Solutions to additive cryptarithms are now trivia with the TABOR-AUTOMATIC CRYPTARITHM SOLVER. This program will accept any cryptarithm involving several additions and one equal sign and solve it in any base.

The program was written as a term project in a course on DATA STRUCTURES. The cryptarithm

$$THREE + NINE = FOUR + EIGHT$$

proved uninteresting in that it has 10 solutions. The replacements to obtain these solutions are:

E	T	R	N	H	U	F	G	O	F
2	3	1	4	0	6	5	7	9	8
2	3	1	4	0	6	5	9	7	8
3	2	4	7	9	1	0	6	8	5
3	2	4	7	9	1	0	8	6	5
3	4	2	5	0	8	6	1	7	9
3	4	2	5	0	8	6	7	1	9
6	7	5	2	0	8	9	1	3	4
6	7	5	2	0	8	9	3	1	4
7	6	8	5	9	3	4	0	2	1
7	6	8	5	9	3	4	2	0	1

Total CPU time was 20 seconds on an XDS Sigma 5. The program is in FORTRAN.

Also solved by Merrill Barnebey, University of Wisconsin at La Crosse; Harold Biller, Brooklyn, New York; Dorothy Brunet, Sherman Oaks, California; Robert Copus, Rose Hulman Institute of Technology; H. Marlon Hewit, Reedley High School, California; J. A. H. Hunter, Toronto, Canada; Janice A. McGoldrick, Cranston High School, Rhode Island; Sam Newman, Atlantic City, New Jersey; Erwin Schmidt, Washington, D. C.; S. O. Shachter, Philadelphia, Pennsylvania; Mary F. Turner, Glen Allen, Virginia; C. S. Venkataraman, Trichur, India; and the proposers.

Solution to Proposal 916:
Mathematics Magazine, 48, (1975), 296.

Trilinear Coordinates

916. [November, 1974] *Proposed by H. Demir, M.E.T.U., Ankara, Turkey.*

Let XYZ be the pedal triangle of a point P with regard to the triangle ABC. Then find the trilinear coordinates x, y, z of P such that

$$YA + AZ = ZB + BX = XC + CY.$$

Solution by M. S. Klamkin, University of Waterloo.

By drawing segments from P parallel to AB and AC respectively and terminating on BC, it follows that

$$BX = x \cot B + z \csc B, \qquad CX = x \cot C + y \csc C.$$

The other distances CY, AY, AZ, BZ follow by cyclic interchange. From the hypothesis,

$$(y+z)(\cot A + \csc A) = (z+x)(\cot B + \csc B) = (x+y)(\cot C + \csc C) = \frac{2s}{3}$$

where s = semiperimeter. Solving:

$$x = \frac{s}{3}\left(\tan\frac{B}{2} + \tan\frac{C}{2} - \tan\frac{A}{2}\right),$$

$$y = \frac{s}{3}\left(\tan\frac{A}{2} + \tan\frac{C}{2} - \tan\frac{B}{2}\right),$$

and

$$z = \frac{s}{3}\left(\tan\frac{A}{2} + \tan\frac{B}{2} - \tan\frac{C}{2}\right).$$

Also solved by D. M. Bailey, Gordon Bennett, Alfred Brousseau, Michael Goldberg, J. M. Stark, and the proposer.

Solution to Proposal 963:
Mathematics Magazine, 50, (1977), 53.

Convex Quadrilaterals **January 1976**

963. Characterize convex quadrilaterals with sides a, b, c, and d such that

$$\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix} = 0.$$

[*Hüseyin Demir, Ankara, Turkey.*]

Solution: It is easy to show, by adding and subtracting rows and columns, that the given determinant equation is equivalent to

$$(a+c+b+d)(a+c-b-d)[(a-c)^2+(b-d)^2]=0.$$

Since we have a, b, c, and d all positive, then either

$$a+c=b+d, \quad \text{or} \quad a=c \quad \text{and} \quad b=d.$$

In the first case the quadrilateral can be circumscribed about a circle: in the second it is a parallelogram. The argument reverses to show that, if the quadrilateral either is a parallelogram or possesses an inscribed circle, then the determinant is zero.

CLAYTON W. DODGE
University of Maine at Orono

Also solved by Gerald Bergum, M. G. Greening (Australia), Daniel Mark Rosenblum, J. M. Stark, and the proposer.

Solution to Proposal 998:
Mathematics Magazine, 51, (1978), 199.

A 120° Triangle **November 1976**

998. Characterize all triangles in which the triangle whose vertices are the feet of the internal angle bisectors is a right triangle. [*Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*]

Solution: Let A', B', C' be the feet of the angle bisectors of angles A, B, C, respectively. Then angle $A'C'B'$ is a right angle iff angle ACB is 120 degrees.

Let a, b, c (a', b', c') be the lengths of sides opposite A, B, C (A', B', C'), respectively. Using the law of cosines and the fact that the angle bisector divides the opposite side in the ratio of the adjacent sides it follows that:

$$(c')^2 = \left(\frac{ab}{a+c}\right)^2 + \left(\frac{ab}{b+c}\right)^2 - 2\left(\frac{ab}{a+c}\right)\left(\frac{ab}{b+c}\right)\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

$$(b')^2 = \left(\frac{ac}{b+c}\right)^2 + \left(\frac{ac}{a+b}\right)^2 - 2\left(\frac{ac}{b+c}\right)\left(\frac{ac}{a+b}\right)\left(\frac{a^2+c^2-b^2}{2ac}\right)$$

$$(a')^2 = \left(\frac{bc}{a+c}\right)^2 + \left(\frac{bc}{a+b}\right)^2 - 2\left(\frac{bc}{a+c}\right)\left(\frac{bc}{a+b}\right)\left(\frac{b^2+c^2-a^2}{2bc}\right)$$

Angle $A'C'B'$ is a right angle iff $(a')^2+(b')^2-(c')^2=0$. But this equation simplifies (after much algebra) to

$$\frac{2abc^2(a^2+b^2-c^2+ab)}{(a+b)^2(a+c)(b+c)} = 0.$$

Thus angle $A'C'B'$ is a right angle iff $a^2+b^2-c^2+ab=0$. But the law of cosines yields $a^2+b^2-c^2+ab=0$ iff angle ACB is $120°$.

JOHN OMAN
University of Wisconsin-Oshkosh

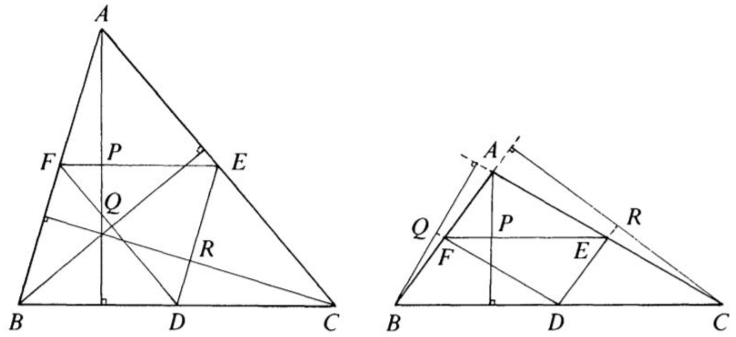
Also solved by Gordon Bennett, Howard Eves, Michael Goldberg, Leonard D. Goldstone, M. G. Greening (Australia), Hubert J. Ludwig, J. M. Stark, Pambuccian Victor (Romania), Robert L. Young, and the proposer.

Solution to Proposal 1197:
Mathematics Magazine, 58, (1985), 240.

Collinear Mid-Altitudes **September 1984**

1197. Characterize the triangles of which the midpoints of the altitudes are collinear. [*Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*]

Solution I: The midpoints of the altitudes of a triangle are collinear if and only if the triangle is right-angled.



Proof. We note first that the altitudes of a triangle all lie inside the triangle if it is acute-angled, while if it has an obtuse angle, two of the altitudes lie outside the triangle.

Let P, Q, and R be the midpoints of the altitudes from the vertices A, B, and C, respectively, of triangle ABC. If D, E, and F are the midpoints of the sides BC, CA, and AB, then P, Q, and R lie, respectively, on EF, FD, and DE, produced if necessary. By Pasch's axiom applied to the triangle DEF, the points P, Q, and R are collinear if and only if two of them coincide with two of D, E, F, in other words lie on the sides of triangle ABC. This occurs if and only if triangle ABC is right-angled.

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University of Cape Town
South Africa

Solution II: The altitudes of a triangle are concurrent at the orthocentre. This is the only property of the altitudes that we need make use of; the answer to the problem is just a special case of the following more general result.

Let D, E, F be points on the side-lines (i.e., lines containing the sides) BC, CA, AB, respectively, of the triangle ABC, such that AD, BE, CF are concurrent at a point P. Then the midpoints of AD, BE, CF are collinear if and only if P coincides with a vertex of triangle ABC or lies on one of its side-lines.

Proof. If P coincides with a vertex, suppose $P = A$ without loss of generality. Then $E = F = A$, and D lies anywhere on the side-line BC; the midpoints of AD, BE, CF are collinear on a line parallel to BC.

If P is not a vertex, we use oblique coordinate axes AB and AC, with suitable units of measurement along the axes so that A, B, C have coordinates $(0,0)$, $(2,0)$, $(0,2)$, respectively. Let P have coordinates (p, q). Then the coordinates of D, E, F are $(2p/(p+q),\ 2q/(p+q))$, $(0, 2q/(2-p))$, $(2p/(2-q), 0)$; we require $p+q \neq 0$, $2-p \neq 0$, $2-q \neq 0$, since otherwise at least one of D, E, F is undefined (for instance, AP is parallel to BC if $p+q=0$). The coordinates of the three midpoints are $(p/(p+q),\ q/(p+q))$, $(1, q/(2-p))$, $(p/(2-q), 1)$; these midpoints are collinear if and only if

$$\begin{vmatrix} p/(p+q) & q/(p+q) & 1 \\ 1 & q/(2-p) & 1 \\ p/(2-q) & 1 & 1 \end{vmatrix} = \frac{2pq(2-p-q)}{(p+q)(2-p)(2-q)} = 0.$$

This occurs if and only if $p=0$ or $q=0$ or $p+q=2$, i.e., if and only if P lies on a side-line of the triangle (in which case two of the midpoints coincide).

Now the orthocentre of a triangle cannot lie on a side-line of the triangle unless it coincides with a vertex, i.e., unless the triangle is right-angled. Hence the midpoints of the altitudes are collinear if and only if the triangle is right-angled.

J. F. RIGBY
University College
Cardiff, Wales

Also solved as in solution I by Jordi Dou (Spain), Howard Eves, Syrous Marivani, Mike Molloy (student, Canada), Richard Parris, Cem Tezer (Turkey), and Michael Woltermann; as in the generalized solution II (but using the theorems of Ceva and Menelaus) by Cem Tezer (Turkey, second solution); using analytic geometry by S. F. Barger, Kenneth Bernstein, Ragnar Dybvik (Norway), Cornelius Groenewoud, Boulkhodra Hacene, L. Kuipers (Switzerland), Hubert J. Ludwig, Bill Olk (student), John Oman, Harry Sedinger, Robert S. Stacy (West Germany), John S. Sumner, Michael Vowe (Switzerland), Jihad Yamout (student), and Robert L. Young; using barycentric or similar coordinate systems by O. Bottema (The Netherlands), J. T. Groenman (The Netherlands, two solutions), J. C. Linders (The Netherlands), and the proposer; using conjugate complex coordinates by Howard Eves (second solution) and Stephanie Sloyan; and using vector analysis by Leonard D. Goldstone and Harry D. Ruderman.

Solution to Proposal 1206:
Mathematics Magazine, 59, (1986), 46.

Sum of Inradii of a Dissected Triangle **January 1985**

1206. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let ABC be a triangle with sides a, b, and c and semiperimeter s. Let the side BC be subdivided using the points $B = P_0, P_1, \ldots, P_{n-1}, P_n = C$ in order. If r_i is the inradius of triangle

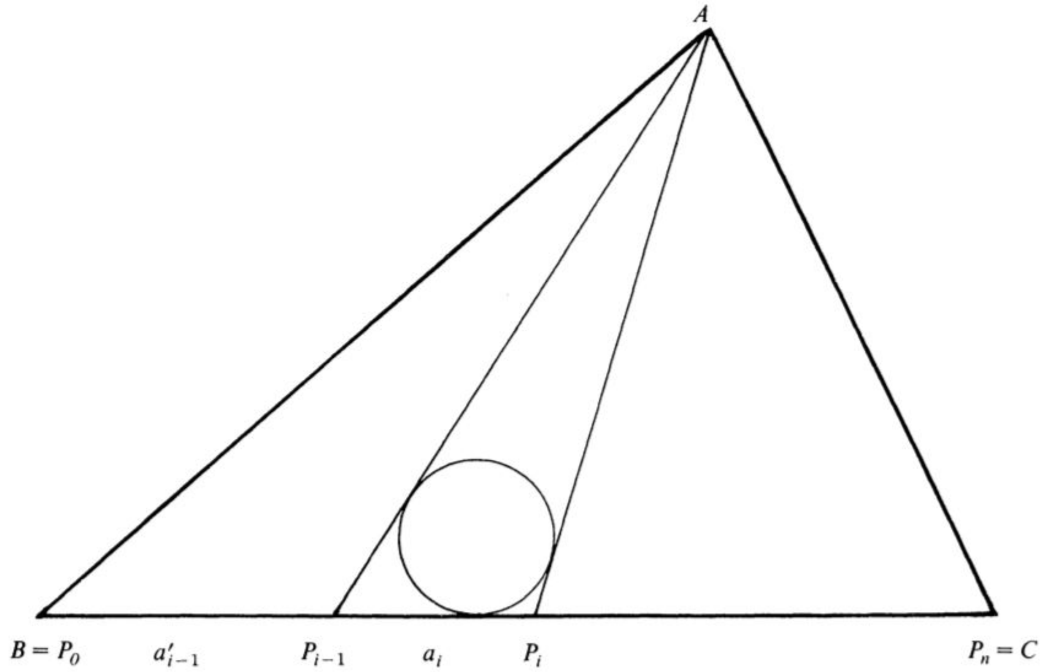


FIGURE 1

$AP_{i-1}P_i$ for $i = 1, \ldots, n$, prove that

$$r_1 + \cdots + r_n < \frac{1}{2} h_a \ln \frac{s}{s-a},$$

where h_a is the length of the altitude from vertex A.

Solution by Vania D. Mascioni, student, ETH Zürich, Switzerland.

For $i = 1, 2, \ldots, n$ let a_i be the base $P_{i-1}P_i$ and s_i the semiperimeter of triangle $AP_{i-1}P_i$, and let a'_i and s'_i be the corresponding quantities for triangle ABP_i. We show below that

$$\frac{s'_{i-1} - a'_{i-1}}{s'_{i-1}} \cdot \frac{s_i - a_i}{s_i} = \frac{s'_i - a'_i}{s'_i} \quad \text{for } 2 \leqq i \leqq n. \tag{1}$$

An easy induction yields

$$\frac{s-a}{s} = \prod_{i=1}^{n} \frac{s_i - a_i}{s_i}.$$

From the arithmetic-geometric mean inequality and the fact that $r_i s_i = \frac{1}{2} a_i h_a$ we obtain

$$\left(\frac{s-a}{s}\right)^{1/n} \leqq \frac{1}{n} \sum_{i=1}^{n} \frac{s_i - a_i}{s_i} = \frac{1}{n} \sum_{i=1}^{n} \left(1 - \frac{a_i}{s_i}\right) = 1 - \frac{2}{nh_a} \sum_{i=1}^{n} r_i,$$

so that

$$\sum_{i=1}^{n} r_i \leqq \frac{nh_a}{2}\left(1 - \left(\frac{s-a}{s}\right)^{1/n}\right),$$

which is stronger than the proposed inequality, which follows if we use $1 - 1/x < \ln x$ for $x > 1$ with $x := (s/(s-a))^{1/n}$.

Proof of (1). To simplify notation, the sides of triangles ABP_{i-1} and $AP_{i-1}P_i$ are relabeled as shown in FIGURE 2. Then (1) becomes

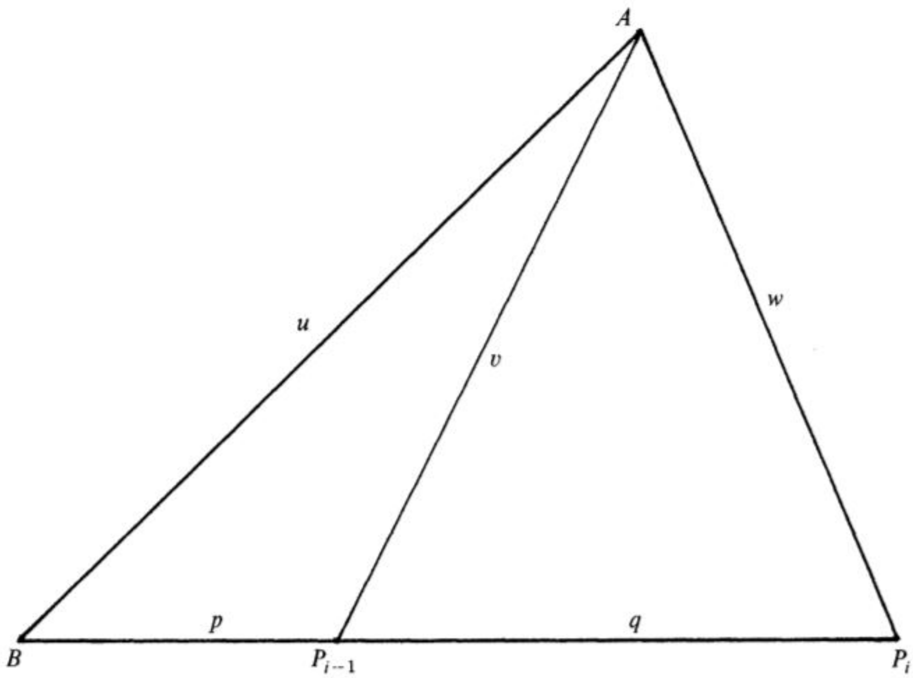


FIGURE 2. **Stewart's theorem.**

$$\frac{u+v-p}{u+v+p} \cdot \frac{v+w-q}{v+w+q} = \frac{u+w-p-q}{u+w+p+q},$$

and an easy (though boring) algebraic manipulation shows this is equivalent to

$$\left(v^2 + p^2 - u^2\right) q + \left(v^2 + q^2 - w^2\right) p = 0.$$

Now by the law of cosines, this is equivalent to

$$2pqv\left(\cos\angle AP_{i-1}B + \cos\angle AP_{i-1}P_i\right) = 0,$$

which is obvious, since $\angle AP_{i-1}B + \angle AP_{i-1}P_i = \pi$. Cf. also Stewart's theorem, in Coxeter and Greitzer, *Geometry Revisited*, p. 6.

Also solved by Jordi Dou (Spain), Václav Konečný & Ronald Shepler, L. Kuipers (Switzerland), Syrous Marivani, William A. Newcomb, Bjorn Poonen (student), J. M. Stark, Paul J. Zwier, and the proposer.

Most solvers used an estimate like

$$\sum_{i=1}^{n} r_i < \sum_{j=1}^{m} r_j' = \sum_{j=1}^{m} \frac{h_a(x_j' - x_{j-1}')}{x_j' - x_{j-1}' + \sqrt{(x_{j-1}')^2 + (h_a)^2} + \sqrt{(x_j')^2 + (h_a)^2}} \approx \int_z^{z+a} \frac{h_a\,dx}{2\sqrt{x^2 + h_a^2}},$$

where $A = (0, h_a)$, $B = (z, 0)$, $C = (z+a, 0)$, $P_j' = (x_j', 0)$, $[P_0', \ldots, P_m']$ is a strict refinement of the partition $[P_0, \ldots, P_n]$ of BC (i.e., each P_i is a P_j', and $m > n$), and r_j' is the inradius of triangle $AP_{j-1}'P_j'$.

Solution to Proposal 1211:
Mathematics Magazine, 59, (1986), 113.

Isoptic of an Ellipse **March 1985**

1211. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*
Find the locus of points under which an ellipse is seen under a constant angle.

Solution by Volkhard Schindler, Berlin, East Germany.
We consider the ellipse $x^2/a^2 + y^2/b^2 = 1$ in a rectangular (x, y) coordinate system. It is well known that the tangent to the ellipse at the point (x_1, y_1) has equation $x_1x/a^2 + y_1y/b^2 = 1$. Since the tangent has x-intercept a^2/x_1 and y-intercept b^2/y_1, the slope m of the tangent from a point (x, y) outside the ellipse is given by

$$m = \frac{y}{x - a^2/x_1} = \frac{y - b^2/y_1}{x},$$

so that

$$\frac{x_1}{a} = \frac{ma}{mx - y} \quad \text{and} \quad \frac{y_1}{b} = \frac{b}{y - mx}. \tag{1}$$

Since (x_1, y_1) lies on the ellipse, we have $[ma/(mx - y)]^2 + [b/(y - mx)]^2 = 1$, which after simplification becomes

$$(x^2 - a^2)m^2 - 2xym + (y^2 - b^2) = 0. \tag{2}$$

If α is the constant angle subtended by the ellipse, then we can number the roots m_1, m_2 of (2) so that $\tan\alpha = (m_1 - m_2)/(1 + m_1m_2)$. Hence

$$\tan^2\alpha = \frac{(m_1 - m_2)^2}{(1 + m_1m_2)^2} = \frac{(m_1 + m_2)^2 - 4m_1m_2}{(1 + m_1m_2)^2},$$

which remains valid if α is replaced by $180° - \alpha$. Since $m_1 + m_2 = 2xy/(x^2 - a^2)$ and $m_1m_2 = (y^2 - b^2)/(x^2 - a^2)$, we obtain

$$\tan^2\alpha = 4\frac{b^2x^2 + a^2y^2 - a^2b^2}{(x^2 + y^2 - a^2 - b^2)^2}. \tag{3}$$

In particular, if $\alpha = 180°$, then (3) reduces to the equation of the original ellipse, as it should. If $\alpha = 90°$, then (3) reduces to $x^2 + y^2 = a^2 + b^2$, which is the equation of a circle of radius $\sqrt{a^2 + b^2}$.

Since equation (3) is not convenient for plotting, we introduce polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$). Then (3) becomes $r^4 - 2Ar^2 + B = 0$, where

$$A = a^2 + b^2 + 2(b^2\cos^2\theta + a^2\sin^2\theta)\cot^2\alpha,$$

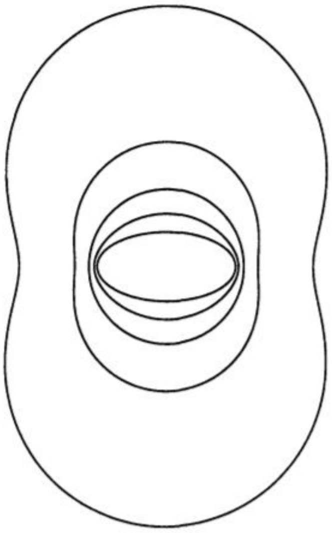
$$B = (a^2 + b^2)^2 + 4a^2b^2\cot^2\alpha,$$

from which we obtain

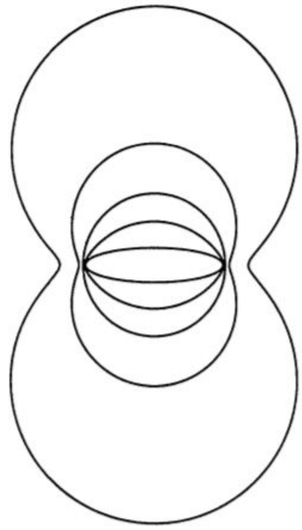
$$r^2 = A \pm \sqrt{A^2 - B}. \tag{4}$$

Since for fixed θ, r^2 decreases as α increases, we see that the plus sign in (4) is used when

$0° < \alpha \leqq 90°$, and the minus sign when $90° \leqq \alpha < 180°$. As seen from the figures, the loci are near-ellipses when $90° < \alpha < 180°$, and are nearly ovals of Cassini or lemniscates of Booth when $0° < \alpha < 90°$. Of course, if $a = b$, all loci are circles.



$a/b = 2$.



$a/b = 4$.

In each figure the values of α for the five curves, starting from the outermost, are 30°, 60°, 90°, 120°, and 180°.

Also solved by Michael V. Finn, J. T. Groenman (The Netherlands), L. Kuipers (Switzerland), Vania Mascioni (student, Switzerland), William A. Newcomb, Richard Parris, Stephanie Sloyan, and Robert L. Young. Solved partially by Zachary Franco (student) and the proposer.

M. S. Klamkin (Canada) found the result in R. C. Yates, *A Handbook on Curves and their Properties*, J. W. Edwards, Ann Arbor, 1947 (reprinted as *Curves and their Properties*, NCTM, 1974), pp. 138–140, where the terms *isoptic* and *orthoptic* are defined. None of the solvers considered the exceptional cases arising when, for example, x, x_1, y, y_1, or m is zero or m is infinite in (1).

Solution to Proposal 1298:
Mathematics Magazine, 62, (1989), 200.

Circumscribable quadrangle **June 1988**

1298. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

A quadrilateral $ABCD$ is circumscribed about a circle, and P, Q, R, S are the points of tangency of sides AB, BC, CD, DA respectively. Let $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DA|$, and $p = |QS|$, $q = |PR|$. Show that

$$\frac{ac}{p^2} = \frac{bd}{q^2}.$$

I. *Solution by J. M. Stark, Lamar University, Texas.*

Denote by r the radius of the circle tangent to the sides of $ABCD$, and let $\alpha, \beta, \gamma, \delta$ be the angles subtended at the center of the circle by the chords SP, PQ, QR, RS respectively.

We have $a = |AP| + |PB|$, $b = |BQ| + |QC|$, $c = |CR| + |RD|$, $d = |DS| + |SA|$ and right triangle geometry gives $|AP| = |SA| = r\tan(\alpha/2)$, $|BQ| = |PB| = r\tan(\beta/2)$, $|CR| = |QC| = r\tan(\gamma/2)$, $|RD| = |DS| = r\tan(\delta/2)$. It follows that

$$ac = r^2(\tan(\alpha/2) + \tan(\beta/2))(\tan(\gamma/2) + \tan(\delta/2)),$$

and

$$bd = r^2(\tan(\beta/2) + \tan(\gamma/2))(\tan(\delta/2) + \tan(\alpha/2)). \tag{1}$$

Application of the identity $\tan(x) + \tan(y) = (\sin(x+y))/(\cos(x)\cos(y))$ to (1) gives

$$\frac{ac}{bd} = \frac{\sin((\alpha+\beta)/2)\sin((\gamma+\delta)/2)}{\sin((\beta+\gamma)/2)\sin((\alpha+\delta)/2)}. \tag{2}$$

From $\alpha + \beta + \gamma + \delta = 2\pi$ we obtain $\sin((\gamma + \delta)/2) = \sin((\alpha + \beta)/2)$ and $\sin((\alpha + \delta)/2) = \sin((\beta + \gamma)/2)$, which, when combined with (2) yields

$$\frac{ac}{bd} = \frac{\sin^2((\alpha+\beta)/2)}{\sin^2((\beta+\gamma)/2)}. \tag{3}$$

Since $p^2 = (2r\sin((\alpha+\beta)/2))^2$ and $q^2 = (2r\sin((\beta+\delta)/2))^2$, it follows from (3) that $ac/bd = p^2/q^2$.

II. *Solution by O. Nouhaud, Faculté des Sciences de Limoges, France.*

Let A', B', C', D' be the inverses of A, B, C, D respectively under the inversion about the inscribed circle with center O and radius r. We know that

$$|A'B'| = r^2\frac{|AB|}{|OA||OB|}$$

(e.g., see *A Survey of Geometry*, Howard Eves, Allyn and Bacon, Boston, 1963, Theorem 3.4.20, p. 153). A circular permutation gives three similar relations. Moreover, $2|A'B'| = |SQ|$ because A' bisects SP and B' bisects PQ. Similarly, $2|C'D'| = |SQ|$ and $2|A'D'| = 2|B'C'| = |RP|$. The desired result follows from these relations.

Also solved by Mangho Ahuja, Wadie A. Bassali (Kuwait), J.-M. Becker (France), Bilkent University Problem Solving Group (Turkey), J. C. Binz (Switzerland), Duane M. Broline, Brown University Fly-Fishing Club, Onn Chan (student), Gang Chang (student), Chico Problem Group, Timothy Chow, Ragnar Dybvik (Norway), E. C. Friedman, Francis M. Henderson, J. Heuver (Canada), Geoffrey A. Kandall, Hans Kappus (Switzerland), Václav Konečný, L. Kuipers (Switzerland), Helen M. Marston, Richard E. Pfiefer, James S. Robertson, Harry D. Ruderman, Raul A. Simon (Chile), László Szücs, R. S. Tiberio, George Vakanas (student), and the proposer.

Solution to Proposal 1305:
***Mathematics Magazine*, 62, (1989), 278.**

Inradii Identity **October 1988**

1305. *Proposed by H. Demir and C. Tezer, Middle East Technical University, Ankara, Turkey.*

Let $P_0 = B, P_1, P_2, \ldots, P_n = C$ be points, taken in that order, on the side BC of the triangle ABC. If r, r_i, and h denote, respectively, the inradii of the triangles ABC, $AP_{i-1}P_i$, and the common altitude, prove that

$$\prod_{i=1}^{n}\left(1 - \frac{2r_i}{h}\right) = 1 - \frac{2r}{h}.$$

Solution by Jim Francis, University of Washington, Seattle, Washington.

It suffices to prove the case where $n = 2$, since the formula then follows by induction.

From Euclidean geometry, we know that the inradius of any triangle is the quotient of its area by its semiperimeter. Hence, if we let $x_1 = BP_1$, $x_2 = P_1C$, $a_1 = AB$, $a_2 = AP_1$, and $a_3 = AC$, then

$$r_1 = \frac{\frac{1}{2}x_1 h}{\frac{1}{2}(x_1 + a_1 + a_2)} = \frac{x_1 h}{x_1 + a_1 + a_2}$$

$$r_2 = \frac{x_2 h}{x_2 + a_2 + a_3},$$

and

$$r = \frac{(x_1 + x_2)h}{x_1 + x_2 + a_1 + a_3}.$$

This implies that

$$\left(1 - \frac{2r_1}{h}\right)\left(1 - \frac{2r_2}{h}\right) = \left(1 - \frac{2x_1 h}{h(x_1 + a_1 + a_2)}\right)\left(1 - \frac{2x_2 h}{h(x_2 + a_2 + a_3)}\right)$$
$$= \frac{(a_1 + a_2 - x_1)(a_2 + a_3 - x_2)}{(a_1 + a_2 + x_1)(a_2 + a_3 + x_2)}.$$

Similarly we have

$$\left(1 - \frac{2r}{h}\right) = \frac{a_1 + a_3 - x_1 - x_2}{a_1 + a_3 + x_1 + x_2}.$$

It remains to show that the right-hand sides of the above two equations are equal, or equivalently, to show that

$$(a_1 + a_2 + x_1)(a_2 + a_3 + x_2)(a_1 + a_3 - x_1 - x_2)$$
$$= (a_1 + a_2 - x_1)(a_2 + a_3 - x_2)(a_1 + a_3 + x_1 + x_2).$$

Expanding and eliminating that which is common to each side, the right side reduces to

$$(-a_1^2 + a_2^2 + x_1^2)x_2 + (-a_3^2 + a_2^2 + x_2^2)x_1,$$

while the left side reduces to the additive inverse of this expression. Thus it remains to show that the above expression is zero. This follows from the law of cosines as follows.

Let $\alpha = \angle AP_1B$. Then

$$-a_1^2 + a_2^2 + x_1^2 = 2a_2 x_1 \cos\alpha$$

while

$$-a_3^2 + a_2^2 + x_2^2 = 2a_2 x_2 \cos(\pi - \alpha) = -2a_2 x_2 \cos\alpha,$$

and the proof is complete.

Also solved by S. Belbas, Francisco Bellot-Rosado (Spain), Anna Boettcher and Václav Konečný, Duane M. Broline, Michael V. Finn, John F. Goehl, Jr., Francis M. Henderson, J. Heuver (Canada), Hans Kappus (Switzerland), L. Kuipers (Switzerland), Lamar University Problem Solving Group, J. C. Linders (The Netherlands), Vania Mascioni (Switzerland), The Oxford Running Club, Werner Raffke (West Germany), John P. Robertson, Hyman Rosen, Volkhard Schindler (East Germany), Michael Vowe (Switzerland), A. Zulauf (New Zealand), and the proposer.

Solution to Proposal 1327:
***Mathematics Magazine*, 63, (1990), 275.**

Diagonals of Exscribed Quadrangles **October 1989**

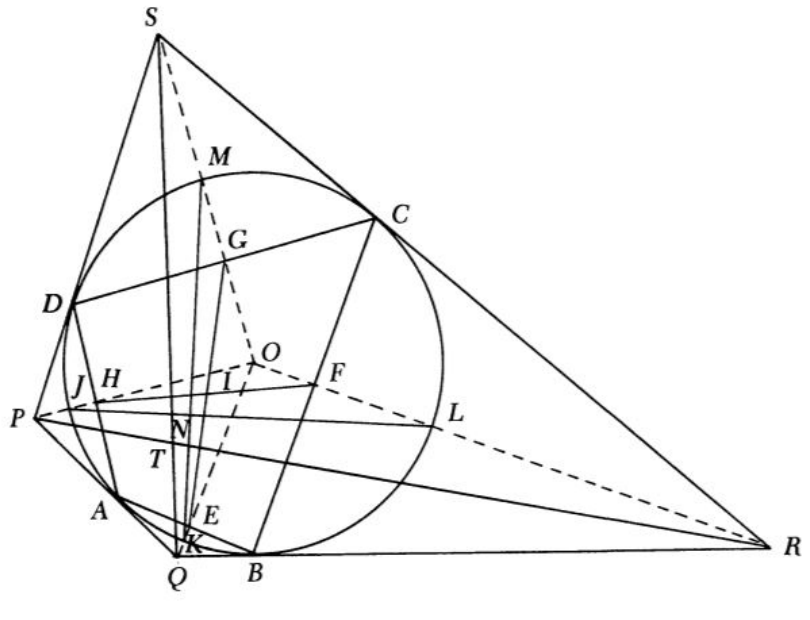
1327. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let the sides PQ, QR, RS, SP of a convex quadrangle $PQRS$ touch an inscribed circle at A, B, C, D and let the midpoints of the sides AB, BC, CD, DA be E, F, G, H. Show that the angle between the diagonals PR, QS is equal to the angle between the bimedians EG, FH.

I. *Solution by Jordi Dou, Barcelona, Spain.*

Let the inscribed circle have radius r and center O. Let J, K, L, M be the intersection points of the circle with the lines OHP, OEQ, OFR, OGS respectively.

Let $N = JL \cap KM$, $T = PR \cap QS$, $I = EG \cap FH$. Note that JL and KM are perpendicular [because the arcs JAK and LCM together comprise half the perimeter of the circle], so [since $\triangle JOL$ is isosceles] KM is parallel to the angle bisector of $\angle JOL$. Also, note that the lines PR, HF are antiparallel with respect to the sides of $\angle JOL$ [that is, $\angle OHF = \angle ORP$ and $\angle OFH = \angle OPR$], because $OH \cdot OP = OF \cdot OR = r^2$, and so $\triangle OHF$, $\triangle ORP$ are similar, [since $OH/OF = OR/OP$]. This [together with the fact that $\triangle JOL$ is isosceles] implies that the lines PR, HF form equal angles (say α) with JL. Similarly, the lines QS, EG form equal angles (say β) with KM. We then have $\angle GIF = 90° - (\alpha + \beta)$ while $\angle RTS = 90° + (\alpha + \beta)$, and we are done.



II. *Solution by Jiro Fukuta, Motosu-gun, Gifu-ken, Japan.*

Let O be the center of the inscribed circle of the quadrangle $PQRS$ and r be the length of the radius. Let P, Q, R, S be denoted by the complex numbers $\alpha, \beta, \gamma, \delta$, respectively, on the complex plane with the origin at O. Then E, F, G, H correspond to $r^2/\overline{\beta}, r^2/\overline{\gamma}, r^2/\overline{\delta}, r^2/\overline{\alpha}$, respectively.

To obtain the conclusion, it is sufficient to prove that

$$F \equiv \left(\frac{\alpha - \gamma}{\beta - \delta}\right) \div \left(\frac{r^2/\overline{\beta} - r^2/\overline{\delta}}{r^2/\overline{\alpha} - r^2/\overline{\gamma}}\right)$$

is real. We have

$$F = \frac{\alpha-\gamma}{\beta-\delta} \cdot \frac{\dfrac{r^2\overline{\gamma} - r^2\overline{\alpha}}{\overline{\alpha}\overline{\gamma}}}{\dfrac{r^2\overline{\delta} - r^2\overline{\beta}}{\overline{\beta}\overline{\delta}}}$$

$$= \frac{\alpha-\gamma}{\beta-\delta} \cdot \frac{\overline{\alpha}-\overline{\gamma}}{\overline{\beta}-\overline{\delta}} \cdot \frac{\overline{\beta\delta}}{\overline{\alpha\gamma}}$$

$$= \frac{|\alpha-\gamma|^2}{|\beta-\delta|^2} \overline{\left(\frac{\delta}{\alpha} \cdot \frac{\beta}{\gamma}\right)}.$$

But $(\delta/\alpha)(\beta/\gamma)$ is real, because $\arg(\delta/\alpha) + \arg(\beta/\gamma) = \arg POS + \arg ROQ = \pi$. This completes the proof.

Also solved by Duane M. Broline, Timothy V. Craine, John F. Goehl, Jr., Francis M. Henderson, Paul Martin, and the proposer.

Solution to Proposal 1356:
Mathematics Magazine, 64, (1991), 278.

Collinearity and symmetry **October 1990**

1356. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let P, Q be points taken on the side BC of a triangle ABC, in the order B, P, Q, C. Let the circumcircles of PAB, QAC intersect at $M(\neq A)$ and those of PAC, QAB at N. Show that A, M, N are collinear if and only if P and Q are symmetric in the midpoint A' of BC.

Solution by Christos Athanasiadis, student, Massachusetts Institute of Technology, Cambridge, Massachusetts.

Let K and L be the points of intersection of the line BC with the lines AM and AN respectively. Suppose that the line BC is the x-axis of a coordinate system with origin B, and let a, p, q, k, and l denote the x-coordinates of C, P, Q, K, and L respectively. The point K is on the radical axis of the circumcircles of PAB and QAC, hence its powers $k(k-p)$ and $(k-q)(k-a)$ with respect to these two circles are equal. It follows that $k = aq/(a+q-p)$. Similarly we have $l = ap/(a+p-q)$, interchanging the roles of p and q. We easily find that $l = k$ if and only if $p + q = a$ and the result follows.

Also solved by Raúl Marin Carrera (student, Mexico), Jordi Dou (Spain), Jiro Fukuta (Japan), Václav Konečný, Alvaro Avila Márquez (student, Mexico), Richard E. Pfiefer, Ioan Sadoveanu, Jyotirmoy Sarkar (student), Seshadri Sivakumar (Canada), John S. Sumner, and the proposer.

Pfiefer *obtained the solution by inverting the figure through a circle with center A.*

Solution to Proposal 1371:
***Mathematics Magazine*, 65, (1992), 133.**

A Triangle Invariant **April 1991**

1371. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

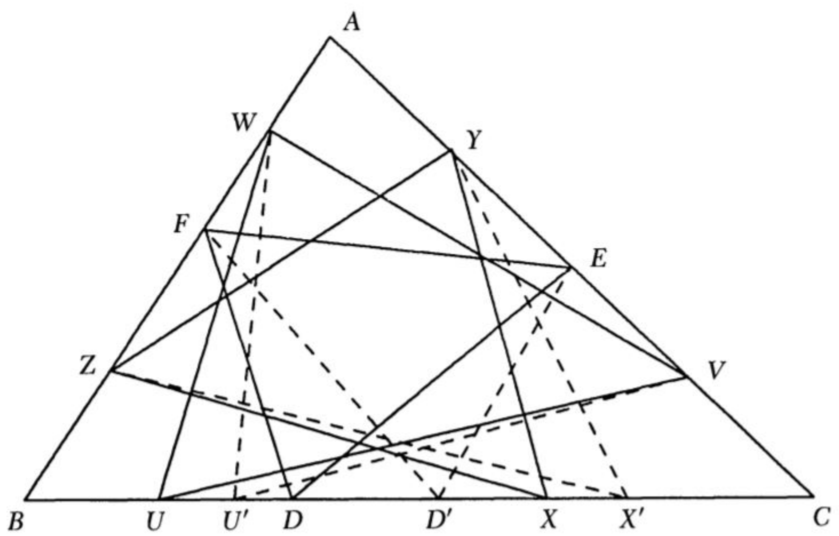
Let A, B, and C be vertices of a triangle and let D, E, and F be points on the sides of BC, AC, and AB, respectively. Let U, X, V, Y, W, Z be the midpoints of, respectively, BD, DC, CE, EA, AF, FB. Prove that

$$\text{Area}(\triangle UVW) + \text{Area}(\triangle XYZ) - \tfrac{1}{2}\text{Area}(\triangle DEF)$$

is a constant independent of D, E, and F.

I. *Solution by Jordi Dou, Barcelona, Spain; submitted on the occasion of his 80th birthday.*

First, let D' be any point between C and D and take U', X' to be the midpoints of $BD', D'C$. Then $XX' = UU' = \frac{1}{2}DD'$.



Let $h_F, h_A, h_W, \ldots$ denote the distances from $F, A, W, \ldots$ to BC respectively. Clearly $h_Z = \frac{1}{2}h_F$, and $h_W = \frac{1}{2}(h_F + h_A)$, and therefore by addition, $h_Z + h_W - h_F = \frac{1}{2}h_A = h_Y + h_V - h_E$. We let $[PQR]$ denote the area of triangle PQR, and set $S = [ABC]$, $\sigma = [DEF]$, $\sigma_1 = [XYZ]$, $\sigma_2 = [UVW]$, $\sigma' = [D'EF]$, $\sigma_1' = [X'YZ]$, and $\sigma_2' = [U'VW]$.

Using these identities, we find that

$$\sigma' - \sigma = \tfrac{1}{2}DD'(h_E - h_F),$$
$$\sigma_1' - \sigma_1 = \tfrac{1}{2}XX'(h_Y - h_Z) = \tfrac{1}{4}DD'(h_Y - h_Z)$$

and

$$\sigma_2' - \sigma_2 = \tfrac{1}{4}DD'(h_V - h_W).$$

It follows that

$$\begin{aligned}(\sigma_1' + \sigma_2' - \tfrac{1}{2}\sigma') - (\sigma_1 + \sigma_2 - \tfrac{1}{2}\sigma) &= \tfrac{1}{4}DD'(h_Y - h_Z + h_V - h_W - h_E + h_F)\\ &= \tfrac{1}{4}DD'((h_Y + h_V - h_E) - (h_Z + h_W - h_F))\\ &= \tfrac{1}{4}DD'(\tfrac{1}{2}h_A - \tfrac{1}{2}h_A)\\ &= 0.\end{aligned}$$

By symmetry, it is clear that the preceding is also 0 when D' is between B and D.

In exactly the same way, taking F' on AB instead of F and triangle $D'EF'$ for $D'EF$, and after this, taking E' on AC instead of E and triangle $D'E'F'$ for $D'EF'$, we find that $\sigma_1 + \sigma_2 - \frac{1}{2}\sigma$ is invariant with respect to DEF.

We obtain the value of $\sigma_1 + \sigma_2 - \frac{1}{2}\sigma$ by putting $E = A$, $F = B$, $D = C$. Then $X = C$, $Y = A$, $Z = B$, U is the midpoint of BC, V is the midpoint of CA, W is the midpoint of AB. Also, $\sigma = S$, $\sigma_1 = S$, $\sigma_2 = (1/4)S$, and therefore, $\sigma_1 + \sigma_2 - \frac{1}{2}\sigma = \frac{3}{4}S$.

II. *Solution by László Szücs, Fort Lewis College, Durango, Colorado.*

We shall use the notation $[ABC] = \text{Area}(\triangle ABC)$. The given expression can be written as

$$\begin{aligned}&([ABC] - ([AWV] + [BUW] + [CVU]))\\ &\quad + ([ABC] - ([AZY] + [BXZ] + [CYX]))\\ &\quad - (1/2)([ABC] - ([AFE] + [BDF] + [CED])).\end{aligned}$$

Using the relations $[AFE] = 4[AWY]$, $[BDF] = 4[BUZ]$, and $[CED] = 4[CVX]$, the expression becomes

$$\begin{aligned}\tfrac{3}{2}[ABC] - ([AWV] &- [AWY] + [AZY] - [AWY]\\ &+ [BUW] - [BUZ] + [BXZ] - [BUZ]\\ &+ [CVU] - [CVX] + [CYX] - [CVX]).\end{aligned}$$

We now use the relation $[AWV] - [AWY] = [VYW] = \frac{1}{4}[CAF]$ and its five analogues to obtain

$$\tfrac{3}{2}[ABC] - \tfrac{1}{4}([CAF] + [EAB] + [ABD] + [FBC] + [BCE] + [DCA]),$$

which is easily seen to equal

$$\tfrac{3}{2}[ABC] - \tfrac{3}{4}[ABC] = \tfrac{3}{4}[ABC].$$

Also solved by Larry E. Askins, Eynshteyn Averbukh, Seung-Jin Bang (Korea), Karen Benbury, Francisco Bellot Rosado (Spain), Scott D. Cohen (student), C. Patrick Collier, Miquel Amengual Covas (Spain), Jordi Dou (Spain), Ragnar Dybvik (Norway), Kao H. and Irene C. Sze, Milton P. Eisner, Jiro Fukuta (Japan), Thomas E. Gantner, John F. Goehl, Jr, Cornelius Groenewoud, H. Guggenheimer, Francis M. Henderson, Ralph P. Grimaldi, Russell Jay Hendel, Paul Irwin, Geoffrey A. Kandall, Vaćlav Konečný, Philip Lau, Eugene Lee, Peter W. Lindstrom, James Pfaendtner, Richard E. Pfiefer, Rolf Rosenkranz (Germany), Ioan Sadoveanu, Jyotirmoy Sarkar, Volkhard Schindler (Germany), Mohammad Parvez Shaikh (student), Ching-Kuang Shene, John S. Sumner, Jordan Tabov (Bulgaria), Michael Vowe, and the proposer.

Tabov proved the more general result. Consider a triangle $A_1A_2A_3$, a real number α different from 0 and 1, and real numbers λ and μ. For arbitrary points X_1, X_2, and X_3 respectively on the lines A_2A_3, A_3A_1, and A_1A_2, define points C_{ij}, $(i, j = 1, 2, 3;\ i \neq j)$ by $\overrightarrow{OC_{ij}} = \alpha\overrightarrow{OA_i} + \beta\overrightarrow{OX_j}$, where O is any point outside the plane of the triangle $A_1A_2A_3$ and $\beta = 1 - \alpha$. Let $F(X_1, X_2, X_3) = \lambda[C_{23}C_{31}C_{12}] + \mu[C_{13}C_{21}C_{32}] - [X_1X_2X_3]$, where the square brackets denote signed area, and X_1, X_2, and X_3 describe independently respectively the lines A_2A_3, A_3A_1 and A_1A_2. Then the function $F(X_1, X_2, X_3)$ is constant if and only if $\lambda = \mu = \frac{1}{2}(1 - \alpha)^{-2}$. (The given problem corresponds to the case $\alpha = 1/2$.)

Solution to Proposal 1377:
***Mathematics Magazine*, 65, (1992), 199.**

A triangle invariant **June 1991**

1377. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let DEF be a variable triangle inscribed in triangle ABC, and let U, X, V, Y, W, Z be the midpoints of the line segments BD, DC, CE, EA, AF, and FB, respectively. Show that the expression

$$|UVW| + |XYZ| - \tfrac{1}{2}|DEF|$$

for areas is constant.

Solution by Hans Kappus, Mathematisches Institut der Universität, Basel, Switzerland.

Denote the expression in question by S. We show that $S = (3/4)$ Area ABC.

Since $S/\text{Area } ABC$ remains unchanged under affine transformations we may choose the affine coordinate system so that $A = (0,0)$, $B = (1,0)$, and $C = (0,1)$. Now let

$$D = (1-r, r), \qquad E = (0, s), \qquad F = (t, 0); \qquad 0 \le r, s, t, \le 1.$$

Then we have

$$U = (1 - r/2, r/2), \qquad V = (0, (1+s)/2), \qquad W = (t/2, 0),$$
$$X = ((1-r)/2, (1+r)/2), \qquad Y = (0, s/2), \qquad Z = ((1+t)/2, 0).$$

Using these coordinates the following areas may be calculated in a straightforward manner:

$$\text{Area } UVW = \tfrac{1}{8}(2 - r + 2s - t - rs + rt - st)$$
$$\text{Area } XYZ = \tfrac{1}{8}(1 + r + t - rs + rt - st)$$
$$\text{Area } DEF = \tfrac{1}{2}(s - rs + rt - st).$$

From this it follows that $S = 3/8 = (3/4)$ Area ABC.

Also solved by Beno Arbel (Israel), H. Guggenheimer, Francis M. Henderson, John G. Heuver (Canada), Thomas Jager, Václav Konečny, Helen M. Marston, Ralph Merrill, José Heber Nieto (Venezuela), Chrysostom G. Petalas (Greece), F. C. Rembis, Robert L. Young, Paul J. Zwier, an unsigned solution, and the proposer.

Several people mentioned that the problem is incorrect as stated. The intention in the problem was that D, E, F should be on line segments BC, CA, and AB respectively. A corrected version of this problem appears as 1371 *in April* 1991, *and several solutions are given in the April* 1992 *issue. Somehow the uncorrected version did not get lifted from the file of accepted proposals, so it inadvertently reappeared. Apologies.*

Solution to Proposal 1405:
Mathematics Magazine, 66, (1993), 269.

Isogonally related circles **October 1992**

1405. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

Two circles inscribed in distinct angles of a triangle are *isogonally related* if the tangents from the third vertex not coinciding with the sides are symmetric with respect to the bisector of the third angle. Given three circles inscribed in distinct angles of a triangle, prove that if any two of the three pairs of circles are isogonally related then so is the third pair.

Solution by the proposer.

Let $\Gamma_1, \Gamma_2, \Gamma_3$ be circles inscribed in angles BAC, CBA, ACB, respectively, of the given triangle ABC. Let I_i, r_i be the center and the radius of Γ_i, $i = 1, 2, 3$. Let E and F denote the points on side AB, $E, F \notin \{A, B\}$, such that CE and CF are tangent to Γ_1 and Γ_2, respectively. Let $\gamma = \angle ACE$ and $\mu = \angle FCB$. As usual, let a, b, c denote the lengths of the sides BC, CA, AB, respectively.

By considering triangles AI_1C and I_2BC, respectively, we have

$$b = \left(\cot\frac{A}{2} + \cot\frac{\gamma}{2}\right)r_1$$

$$a = \left(\cot\frac{B}{2} + \cot\frac{\mu}{2}\right)r_2.$$

Now Γ_1 and Γ_2 are isogonally related if, and only if, $\gamma = \mu$, and, using the previous equations, this is the case if and only if

$$\frac{b}{r_1} - \cot\frac{A}{2} = \frac{a}{r_2} - \cot\frac{B}{2},$$

or equivalently,

$$\frac{b}{r_1} - \frac{s-a}{r} = \frac{a}{r_2} - \frac{s-b}{r},$$

where $2s = a + b + c$ and r is the inradius of triangle ABC. This can be regrouped into the form

$$\frac{a}{(1/r_1 - 1/r)} = \frac{b}{(1/r_2 - 1/r)}. \tag{1}$$

Similarly, Γ_2 and Γ_3 are isogonally related if, and only if,

$$\frac{b}{(1/r_2 - 1/r)} = \frac{c}{(1/r_3 - 1/r)}. \tag{2}$$

Combining (1) and (2), we obtain

$$\frac{a}{(1/r_1 - 1/r)} = \frac{c}{(1/r_3 - 1/r)},$$

which happens if, and only if, Γ_1 and Γ_3 are isogonally related.

Also solved by Richard Holzsager, Jiro Fukuta (Japan), and Francisco Bellot Rosado and María Ascensión López (Spain).

8 Quickies for *Mathematics Magazine*

List of Quicky problems composed by Hüseyin Demir

[1] Quicky 117, *Mathematics Magazine*, 28, (1954-1955), 37.
[2] Quicky 138, *Mathematics Magazine*, 28, (1954-1955), 241.
[3] Quicky 141, *Mathematics Magazine*, 28, (1954-1955), 292.
[4] Quicky 166, *Mathematics Magazine*, 29, (1955-1956), 29.
[5] Quicky 188, *Mathematics Magazine*, 30, (1956-1957), 172.
[6] Quicky 227, *Mathematics Magazine*, 32, (1958-1959), 32.
[7] Quicky 234, *Mathematics Magazine*, 32, (1958-1959), 113.
[8] Quicky 242, *Mathematics Magazine*, 32, (1958-1959), 229.
[9] Quicky 266, *Mathematics Magazine*, 33, (1959-1960), 302.
[10] Quicky 281, *Mathematics Magazine*, 34, (1961), 303.
[11] Quicky 284, *Mathematics Magazine*, 34, (1961), 303.
[12] Quicky 341, *Mathematics Magazine*, 37, (1964), 251.
[13] Quicky 343, *Mathematics Magazine*, 37, (1964), 251.
[14] Quicky 710, *Mathematics Magazine*, 59, (1986), 112.

Quicky 117, *Mathematics Magazine*, 28, (1954-1955), 37.

Q 117 How many squares are there on a chessboard? [*Submitted by Huseyin Demir*]

Quicky 138, *Mathematics Magazine*, 28, (1954-1955), 241.

Q 138. Find the sum $S_m = 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$ [*Submitted by Huseyin Demir*].

Quicky 141, *Mathematics Magazine*, 28, (1954-1955), 292.

Q 141. If the twelve months of the year are written in the order offered by $5n + 2 \pmod{12}$, $n = 1, 2, 3, \ldots 12$, what can be said about the characteristics of the first seven, the next four, and of the last month? [*Submitted by Huseyin Demir.*]

Quicky 166, *Mathematics Magazine*, 29, (1955-1956), 29.

Q 166. Can the sum of the cubes of the first m consecutive integers

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Quicky 188, *Mathematics Magazine*, 30, (1956-1957), 172.

Q 188. At what times must the hands of a clock be interchanged in order to obtain new correct time? [*Submitted by Huseyin Demir*].

Quicky 227, *Mathematics Magazine*, 32, (1958-1959), 32.

Q 227. Find a function $f(n)$ such that $f(1)$, $f(2)$, $f(3)$, ..., $f(13)$, $f(14)$ **be**

Quicky 234, *Mathematics Magazine*, 32, (1958-1959), 113.

Q 234. If the sum of the coefficients of $A(x)B(x)$ is zero, the sum of the coefficients of one of the polynomials is necessarily zero. [*Submitted by Huseyin Demir*]

Quicky 242, *Mathematics Magazine*, 32, (1958-1959), 229.

Q 242. Find an $f(n)$ such that $f(\text{even}) = \frac{1}{2}$ and $f(\text{odd}) = 1$ [*Submitted by Huseyin Demir*]

Quicky 266, *Mathematics Magazine*, 33, (1959-1960), 302.

Q 266. If p is a prime number greater than 3, then $p^2 + 2$ is composite. [*Submitted by Huseyin Demir*]

Quicky 281, *Mathematics Magazine*, 34, (1961), 303.

Q 281. Evaluate the radius of the inner tritangent circle to excircles of a triangle [*Submitted by Huseyin Demir*].

Quicky 284, *Mathematics Magazine*, 34, (1961), 303.

Q 284. What is the locus of points whose projections on the sides of a triangle are collinear? [*Submitted by Huseyin Demir*].

Quicky 341, *Mathematics Magazine*, 37, (1964), 251.

Q341. Find the limit of the fraction as n approaches infinity:

$$\frac{\phi(1) + \phi(2) + \phi(3) + \cdots + \phi(n)}{1 + 2 + 3 + \cdots + n}$$

where $\phi(n)$ is Euler's totient.

[*Submitted by Huseyin Demir.*]

Quicky 343, *Mathematics Magazine*, 37, (1964), 251.

Q343. Identify the angle θ satisfying

$$\frac{\sin(1\cdot\theta + 15°)}{\sqrt{1}} = \frac{\sin(2\cdot\theta + 15°)}{\sqrt{2}} = \frac{\sin(3\cdot\theta + 15°)}{\sqrt{3}}\,.$$

[*Submitted by Huseyin Demir.*]

Quicky 710, *Mathematics Magazine*, 59, (1986), 112.

Q710. *Submitted by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

If n is any positive integer, show that the number $T = (1/8)n(n+1)(n+2)(n+3)$ is a triangular number.

Solution to Quicky 117:
Mathematics Magazine, 28, (1954-1955), 37.

A 117. Not 64, but $1^2 + 2^2 + \cdots + 8^2 + \dfrac{8 \cdot 9 \cdot 17}{6} = 204$.

Solution to Quicky 138:
Mathematics Magazine, 28, (1954-1955), 241.

A 138.

$$\begin{aligned} S &= \sum_{p=1}^{n} p \cdot p! = \sum_{p=1}^{n} (p+1-1) \cdot p! \\ &= \sum_{p=1}^{n} (p+1)! - \sum_{p=1}^{n} p! \\ &= (n+1)! - 1 \end{aligned}$$

Solution to Quicky 141:
Mathematics Magazine, 28, (1954-1955), 292.

A 141. The first seven months listed will have 31 days, the next four months will have 30 days and the last one has 29 or 28 days.

Solution to Quicky 166:
Mathematics Magazine, 29, (1955-1956), 29.

A 166. If $1^3 + 2^3 + \ldots + m^3 = (m+1)^3 + \ldots + (m+n)^3$

then $2(1^3 + 2^3 + \ldots + m^3) = 1^3 + 2^3 + \ldots + (m+n)^3$

or $2[\tfrac{1}{2}m(m+1)]^2 = [\tfrac{1}{2}(m+n)(m+n+1)]^2$

so $\tfrac{1}{2}m(m+1)\sqrt{2} = \tfrac{1}{2}(m+n)(m+n+1)$

But this last equation is impossible. Therefore the answer is no.

Solution to Quicky 188:

Mathematics Magazine, 30, (1956-1957), 172.

A 188. Let p denote the number of hours and α the fraction of an hour at the time T. When the hands are interchanged we obtain new time t', the corresponding numbers being p', α' (We may suppose $p' \geq p$). At t the angle in hours $p' + \alpha'$ of minute hand is 12 times α:

$$p' + \alpha' = 12, \quad 0 \leq p' < 12, \qquad \alpha' < 1.$$

$$p + \alpha = 12\alpha', \quad 0 \leq p < 12, \qquad \alpha < 1.$$

We have

$$\alpha = (12p' + p)/143$$

$$\alpha' = (p' + 12p)/143 .$$

Hence, given any two positive integers p, p' less than 12 we get α and α', and therefore the required times.

Solution to Quicky 227:
Mathematics Magazine, 32, (1958-1959), 32.

A 227. $f(n) = 1 - \phi(n)$

Solution to Quicky 234:
Mathematics Magazine, 32, (1958-1959), 113.

A 234. For $x = 1$ we have $\Sigma a_i \cdot \Sigma b_i = \Sigma c_i$ and the result follows.

Solution to Quicky 242:
Mathematics Magazine, 32, (1958-1959), 229.

A 242. Construct $f(n)$ so that $f(n) = \frac{1}{2}(n+1-2[n/2])$ where $[n/2]$ is the largest integer in $n/2$.

Solution to Quicky 266:
Mathematics Magazine, 33, (1959-1960), 302.

A 266. If p is a prime exceeding 3 then we have $p^2 + 2 = (6m \pm 1)^2 + 2 \equiv 0 \pmod 3$.

Solution to Quicky 281:

Mathematics Magazine, 34, (1961), 303.

A281. This circle, being the nine-point circle of the triangle, has radius $\frac{1}{2}R$.

Solution to Quicky 284:
Mathematics Magazine, 34, (1961), 303.

A284. If the points are restricted to lie on the plane of the triangle, the locus is the circumcircle of the triangle. Since no such restriction is made, the locus is the right cylinder having this circumcircle as section.

Solution to Quicky 341:
Mathematics Magazine, 37, (1964), 251.

A341. $\phi(k)$ denotes the number of integers smaller than k and prime to it. Hence,

$$\sum_1^n \phi(k)$$

is the total number of relatively prime pairs among the first n integers, the total number of pairs being $\binom{n}{2}$. The limit of the given fraction being

$$\sum_1^n \phi(k) \Big/ \binom{n}{2}$$

it will be the probability that any two integers taken at random be relatively prime. This probability is known to have the value $6/\pi^2$. Hence, the limit of the given fraction is $6/\pi^2$.

Solution to Quicky 343:
Mathematics Magazine, 37, (1964), 251.

A343. The angle θ is evidently $15°$.

Solution to Quicky 710:
Mathematics Magazine, 59, (1986), 112.

A710. T is a triangular number if for some positive integer k, one has $(1/8)\ n(n+1)\ (n+2)\ (n+3) = (1/2)\ k(k+1)$. By considering parity and noting that $n(n+3) < (n+1)\ (n+2)$, one is led to try $k = n(n+3)/2$. Then

$$k+1 = \frac{n(n+3)}{2} + 1 = \frac{(n+1)(n+2)}{2},$$

and the result follows.

10 Contributed Solutions to *Mathematics Magazine*

List of solutions sent to Proposals by Hüseyin Demir

[1] Proposal 192, *Mathematics Magazine*, 28, (1954-1955), 36.
[2] Proposal 203, *Mathematics Magazine*, 28, (1954-1955), 163.
[3] Proposal 204, *Mathematics Magazine*, 28, (1954-1955), 165.
[4] Proposal 221, *Mathematics Magazine*, 28, (1954-1955), 291.
[5] Proposal 226, *Mathematics Magazine*, 29, (1955-1956), 488.
[6] Proposal 270, *Mathematics Magazine*, 30, (1956-1957), 108.
[7] Proposal 272, *Mathematics Magazine*, 30, (1956-1957), 166.
[8] Proposal 332, *Mathematics Magazine*, 32, (1958-1959), 52.
[9] Proposal 353, *Mathematics Magazine*, 32, (1958-1959), 226.
[10] Proposal 374, *Mathematics Magazine*, 33, (1959-1960), 113.
[11] Proposal 383, *Mathematics Magazine*, 33, (1959-1960), 228.
[12] Proposal 387, *Mathematics Magazine*, 33, (1959-1960), 233.
[13] Proposal 400, *Mathematics Magazine*, 34, (1960-1961), 53.
[14] Proposal 401, *Mathematics Magazine*, 34, (1960-1961), 55.
[15] Proposal 646, *Mathematics Magazine*, 40, (1967), 226.
[16] Proposal 653, *Mathematics Magazine*, 42, (1969), 283.
[17] Proposal 1199, *Mathematics Magazine*, 58, (1985), 243.
[18] Proposal 1256, *Mathematics Magazine*, 61, (1988), 54.

Contributed Solution to Proposal 192:
Mathematics Magazine, 28, (1954-1955), 36.

192. [January 1954] *Proposed by V. Thebault, Tennie, Sarthe, France.*

If A', B', C' are the symmetries of the vertices of a triangle ABC with respect to a fixed point, the circumcircles of the three triangles $AB'C'$, $BC'A'$, $CA'B'$ have a point in common which lies on the circumcircle of the triangle ABC.

Solution by Huseyin Demir, Zonguldak, Turkey. It will suffice to prove that any two of the circumcircles intersect on the circumcircle (O) of ABC. Let I be the intersection of the circles $BC'A'$ and $CA'B'$. To prove that it belongs to (O) we show that $\angle BIC = \angle BAC = \angle A$: $\angle BIC = \angle BIA' + \angle A'IC = \angle BC'A' + \angle A'B'C = \angle B'CA + \angle KB'C = \angle B'CK + \angle KB'C = \angle A'KC = \angle BAC = \angle A$.

The first equality follows from the fact that the points B, I, C', A' on one hand and B, B', C, A' on the other lie on the respective circumcircles, and the other equalities from the parallelisms:

$$BC' \parallel CB', A'C' \parallel AC, A'B' \parallel AB.$$

Also solved by *H. E. Fettis, Dayton, Ohio; O. J. Ramler, Catholic University of America and the proposer.*

Contributed Solution to Proposal 203: *Mathematics Magazine*, 28, (1954-1955), 163.

203. [May 1954] *Proposed by Norman Anning, Alhambra, California.*

Prove that three of the intersections of $x^2 - y^2 + ax + by = 0$ and $x^2 + y^2 - a^2 - b^2 = 0$ trisect the circle through these three points.

II. *Solution by Huseyin Demir, Zonguldak, Turkey.* Set $r^2 = a^2 + b^2$ and let the value of y obtained by adding together the two equations be substituted in the first equation. We get an equation:

$$4x^4 + 4ax^3 - 3r^2x^2 - 2ar^2x + a^2r^2 = 0,$$

of fourth degree in x of which the roots are x_1, x_2, x_3, x_4.

If the triangle $A_1A_2A_3$ corresponding to x_1, x_2, x_3 is equilateral, $x_1 + x_2 + x_3$ will vanish (for $A_1A_2A_3$ is in the circle $x^2 + y^2 - r^2 = 0$ centered at 0), and x_4 is from the second coefficient $(x_1 + x_2 + x_3) + x_4 = x_4 = -a$.

Therefore to prove the statement it will suffice to show that the above equation is divisible by $x + a$ and that in the quotient obtained the term x^2 is missing.

By division we get

$$4x^4 + 4ax^3 - 3r^2x^2 - 2ar^2x + a^2r^2 = (x + a)(4x^3 - 3r^2x + ar^2),$$

and this is in agreement with what we said above. Hence $A_1A_2A_3$ is an equilateral triangle.

Contributed Solution to Proposal 204:
***Mathematics Magazine*, 28, (1954-1955), 165.**

204. [May 1954] *Proposed by C. W. Trigg, Los Angeles City College.*

In the triangle ABC let the feet of the median (m_a), of the internal angle bisector (t_a), of the cevian (p_a) to the contact point of the incircle with a, and of the cevian (q_a) to the contact point of the excircle relative to A with a be respectively A_m, A_t, A_p and A_q. Use similar notation for the corresponding lines to b and c.

1). Determine the relationship between the sides of the triangle if the following triads are to be concurrent: p_a, m_b, t_c at S; p_z, q_b, m_c at R; m_a, p_b, t_c at T; q_a, p_b, m_c at V.

2). Show that A_pB_q and A_qB_p are parallel to AB; C_tB_p and SV are parallel to BC; and C_tA_p and RT are parallel to AC.

Solution by Huseyin Demir, Zonguldak, Turkey. 1). We determine the positions of the cevians u_a, v_b, w_c or their feet A_u, B_v, C_w on the respective sides BC, CA, AB by the ratios:

$$k(A_u) = A_uB/A_uC, \quad k(B_v) = B_vC/B_vA, \quad k(C_w) = C_wA/C_wB$$

Since these points are interior points of the sides all these ratios are negative. Their values are tabulated below:

$$k(A_m) = -1,\ k(A_t) = -c/b,\ k(A_p) = -(s-b)/(s-c),\ k(A_q) = -(s-c)/(s-b)$$

$$k(B_m) = -1,\ k(B_t) = -a/c,\ k(B_p) = -(s-c)/(s-a),\ k(B_q) = -(s-a)/(s-c)$$

$$k(C_m) = -1,\ k(C_t) = -b/a,\ k(C_p) = -(s-a)/(s-b),\ k(C_q) = -(s-b)/(s-a)$$

Now, the required common condition is obtained by applying Ceva's theorem to the triples of cevians:

TRIPLES:	POINTS:	CEVA THEOREM:	CONDITIONS:
p_a, m_b, t_c	S	$[-(s-b)/(s-c)][-1][-b/a]=-1$	$(s-b)/(s-c)=a/b$
p_a, q_b, m_c	R	$[-(s-b)/(s-c)][-(s-a)/(s-c)][-1]=-1$	$(s-a)(s-b)=(s-c)^2$
m_a, p_b, t_c	T	$[-1][-(s-c)/(s-a)][-b/a]=-1$	$(s-c)/(s-a)=a/b$
q_a, p_b, m_c	V	$[-(s-c)/(s-b)][-(s-c)/(s-a)][-1]=-1$	$(s-c)^2=(s-a)(s-b)$

These four conditions just obtained are easily seen to be identical with the unique condition

$$c = (a^2 + b^2)/(a + b) .$$

2) (a): To prove $A_pB_q//A_qB_p//AB$ we see that $k(A_p) = 1/k(B_q)$, $k(A_q) = 1/k(b_p)$.

(b): To prove $C_tB_p//BC$ we similarly see $k(B_p) = 1/k(C_t)$ (see cond (3)).

Now to prove $SV//BC$ we apply the Menelaus theorem to the triangles BCB_m, BCC_m cut respectively by the lines ASA_p, AVA_q:

$$(A_pB/A_pC)(AC/AB_m)(SB_m/SB) = 1, \quad \text{then} \quad SB/SB_m = 2k(A_p) .$$

$$(A_qB/A_qC)(VC/VC_m)(AC_m/AB) = 1, \quad \text{then} \quad VC/VC_m = 2/k(A_q).$$

Hence

$$SB/SB_m = 2k(A_p) = 2/k(A_q) = VC/VC_m .$$

This proves that S, V divide BB_m, CC_m in the same ratio. But having $B_mC_m//BC$ the property follows.

(c): To prove $C_tA_p//AC$ we see that $k(C_t) = 1/k(A_p)$.

Then finally to show $RT//AC$ we again apply the Menelaus theorem to the triangles CAC_m, CAA_m cut by the lines BRB_q, BTB_q respectively.

$$k(B_q)\ (BA/BC_m)(RC_m/RC) = 1 \quad \text{then} \quad RC/RC_m = 2k(B_q) ,$$

$$k(B_p)\ (TA/TA_m)(BA_m/BC) = 1 \quad \text{then} \quad TA/TA_m = 2/k(B_p),$$

and

$$RC/RC_m = 2k(B_q) = 2/k(B_p) = TA/TA_m .$$

Hence R and T divide CC_m, AA_m in the same ratio. But having $C_mA_m//CA$ we also have $RT//CA$.

Q. E. D.

Also solved by Sister M. Stephanie, Georgian Court College, N. J. and the proposer.

Contributed Solution to Proposal 221:
Mathematics Magazine, 28, (1954-1955), 291.

221. [November 1954] *Proposed by E.P.Starke, Rutgers University.*

On a conical surface there is traced a spiral which crosses each of the linear elements at a fixed angle ψ. Find a simple expression for the length of this spiral between any two of its points.

Solution by Huseyin Demir, Zonguldak, Turkey.

The cone is a developable surface. When developed the ψ-spiral on the cone is transformed into a ψ-logarithmic spiral on the plane, of which the polar equation is:

$$r = a\,e^{(\cot\psi)\theta}$$

Then $$ds = \sqrt{dr^2 + r^2 d\theta^2} = \frac{a}{\sin\psi}\,e^{(\cot\psi)\theta}\,d\theta.$$

Between two points on the spiral

$$s = \frac{a}{\sin\psi}\int_a^b e^{(\cot\psi)\theta} d\theta = \frac{a}{\cos\psi}\left| e^{(\cot\psi)\theta}\right|_a^b = \frac{\left| \mathrm{r}\right|_a^b}{\cos\psi}$$

$s = (b - a)/\cos\psi$ where a and b denote the distances of the points from the vertex of the cone.

Also solved by Walter B. Carver, Cornell University; M. S. Klamkin, Polytechnic Institute of Brooklyn; S. H. Sesskin, Hofstra College, New,York; A.Sisk, Maryville College, Tennessee and the proposer.

Contributed Solution to Proposal 226:
Mathematics Magazine, 29, (1955-1956), 48.

226. [January 1955] *Proposed by P. D. Thomas, Eglin Air Force Base, Florida.*

Tangents are drawn from a point P to an ellipse. If R and Q are the points of contact and O is the center of the ellipse, find the locus of P if the area of the the quadrilateral $PQOR$ remains constant.

11. *Solution by Huseyin Demir, Zonguldak, Turkey.* The ellipse is an orthogonal projection of a circle. Let $P'Q'OR'$ be the corresponding quadrangle. The locus of P' is a concentric circle, for the two quadrangles are in a constant ratio (in area). Hence the locus of P, projection of P', is an ellipse homothetic with the original one.

Also solved by M. S. Klamkin, Brooklyn Polytechnic Institute; S. H. Sesskin, Hofstra College; E. P. Starke, Rutgers University; Chih-yi Wang, University of Minnesota and the proposer.

Contributed Solution to Proposal 270:
Mathematics Magazine, 30, (1956-1957), 108.

Circles In A Crescent

270. [March 1956] *Proposed by Leon Bankoff, Los Angeles, California.*

A maximum circle is inscribed in a crescent formed by a semicircle and a quadrant of a circle. Find a general expression for the radii of consecutively tangent circles touching the sides of the crescent, the first touching the maximum circle, the second touching the first and so on.

Solution by Huseyin Demir, Kandilli, Bolgesi, Turkey. Let the given circles (0), $(0')$ intersect each other at A and B, and let the center and radius of the nth circle be denoted by (0_n), r_n respectively.

We invert the figure with center at A, $k^2 = AB^2 = 4R^2$ being the power. Under the inversion, (0) is inverted into its tangent line BH_0, and $(0')$ into the line $B0'$, forming an angle of $2\alpha = 45°$. The circles (Ω_i), inverse of (0_i), from a series of tangent circles inscribed in the above angle. Let (Ω_n) touch BH_0 at H_n. Then we may easily find that the radius $\rho_n = \Omega_n H_n$ of (Ω_n) is given by

$$\rho_n = \left(\frac{1 - \sin\alpha}{1 + \sin\alpha}\right)^n \rho_0$$

where

$$\rho_0 = \Omega_0 H_0 = BH_0 \tan\alpha = 2R \tan\alpha.$$

Drawing the common tangent AT_nT_n' to the inverse circles (0_n), (Ω_n) we write

$$r_n = AT_n\ \rho_n/AT_n' = AT_n \cdot AT_n' \rho_n/AT_n'^2 = k^2\rho_n/(A\Omega_n^2 - n^2)$$

Denoting the projection of Ω_n on AB by K_n we have

$$\begin{aligned} A\Omega_n^2 - \rho_n^2 &= AK_n^2 + K_n\Omega_n^2 - \rho_n^2 \\ &= (2R + \rho_n)^2 + BH_n^2 - \rho_n^2 \\ &= 4R^2 + 4R\rho_n + (\rho_n \cot\alpha)^2 \end{aligned}$$

$$r_n = \frac{R^2\rho_n \tan\alpha}{4R^2 + 4R\rho_n + \rho_n^2 \cot^2\alpha}$$

Substituting the value of ρ_n in the above expression we arrive at the desired result, namely

$$r_n = \frac{R}{1 + \frac{1}{2}\left[(1 + \sin\pi/8)^{2n} + (1 - \sin\pi/8)^{2n}\right] \sec^{2n}\pi/8 \cot\pi/8}$$

Also solved by J.W. Clawson, Collegeville, Pennsylvania and the proposer.

Contributed Solution to Proposal 272:
Mathematics Magazine, 30, (1956-1957), 166.

273. [May 1956] *Proposed by N.A. Court, University of Oklahoma.*

The points of intersection of the tangents to the circumcircle of a triangle drawn at the ends of one side is collinear with the two points which that circle marks on the median issued from the opposite vertex and on the parallel through that vertex to the side considered.

II. *Solution by Huseyin Demir, Kandilli, Bolgesi, Turkey.* Let the median and exmedian relative to the vertex A intersect the circumcircle at E, F respectively, and let K_a be the intersection of the tangents at the other vertices B, C. From the harmonic ratios

$$A\ (B,\ C,\ E,\ F) = (AB,\ AC,\ AE,\ AF) = -1$$

Contributed Solution to Proposal 332:
Mathematics Magazine, 32, (1958-1959), 52.

332. [January 1958] *Proposed by Norman Anning, Alhambra, California.*
Prove that there is no polynomial of degree 22 which is an exact divisor of $x^{45}+1$.

II *Solution by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.* The greatest degree of an exact factor is necessarily the number of imprimitive roots of the given equation of which the roots are all distinct. Since the number $45-\phi(45) = 45-24 = 21$ of the imprimitive roots is less than 22, there will be no such a factor.

Also solved by D.A.Breault, Station Hospital, Fort Monmouth, New Jersey; C.F.Pinzka, University of Cincinnati; Norman Anning, Alhambra, California and the proposer. One incorrect solution was received.

Contributed Solution to Proposal 353:
***Mathematics Magazine*, 32, (1958-1959), 226.**

Tangent Circles

353. [September 1958] *Proposed by Karl M. Herstein, New York City, New York.*

Given a line and two points not on the line. Construct two equal circles whose centers are on the given line, which pass through the given points and are tangent to each other.

Solution by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey. Let A, B, and d be the given points and the line. We distinguish two cases:

(1) The circles touch each other externally. Since the radii are equal there are no solutions except when:

(a) The circles coincide. The coincident circles contain both A and B and the center is the intersection of d and the medial line of AB.

(b) The point L of tangency is at infinity: In that case the solution consists of the perpendiculars to d from A and B.

(2) The circles touch each other internally. The solutions, if they exist, must be different from (1a) and (1b).

Take d as the x-axis and let $A(-u, a)$, $B(u, b)$ and $L(\lambda, 0)$. The circles contain the reflections of A, B with respect to d and we may there-suppose $a \geqq b > 0$.

Let the circles intersect d at $A'(\alpha, 0)$, $L(\lambda, 0)$ and L, $B'(\beta, 0)$. We have from the right triangles $A'AL$, LBB':

$$a^2 = (\lambda+u)(-u-\alpha) \qquad b^2 = (\beta-u)(u-\lambda)$$

$$\alpha = -\frac{a^2}{\lambda+u} - u \qquad \beta = -\frac{b^2}{\lambda-u} + u$$

Equating the diameters $(\lambda-\alpha)$ and $(\beta-\lambda)$ we get a cubic equation

$$2\lambda^3 + (a^2+b^2-2u^2)\lambda - u(a^2-b^2) = 0$$

Substituting $a^2 - b^2 = 2c^2$, it reduces to

$$\lambda^3 + (b^2+c^2-u^2)\lambda - uc^2 = 0$$

There are one, two (equal), or three solutions according as the discriminant Δ is positive, zero, or negative.

Now we find the relation for which

$$\Delta = 4p^3 + 27q^2 = 4(b^2+c^2-u^2)^3 + 27u^2c^4 \leqq 0$$

where $p = b^2 + c^2 - u^2$ is necessarily not positive. Hence,

$$b^2 + c^2 \leqq u^2 \quad \text{or} \quad b^2 + c^2 = u^2\cos^2 t \qquad \text{where } 0 \leqq t < \tfrac{1}{2}\pi$$

$$\Delta = 4(u^2\cos^2 t - u^2)^3 + 27u^2c^4 \leqq 0$$

$$-4u^6\sin^6 t + 27u^2c^4 \leqq 0$$

$$27c^4 \leqq 4u^4\sin^6 t$$

Since the quantities are not negative

$$\sqrt{27}\, c^2 \leqq 2u^2\sin^3 t \leqq 2u^2$$

We have finally

$$\sqrt{27}\sqrt{a^2-b^2} \begin{array}{ll} > 2u & \text{one real root} \\ = 2u & \text{double or triple root} \\ < 2u & \text{three real roots} \end{array}$$

Also solved by Sam Kravitz, East Cleveland, Ohio.

Contributed Solution to Proposal 374:
Mathematics Magazine, 33, (1959-1960), 113.

Equivalent Triangles

374. [March 1959] *Proposed by Victor Thebault, Tennie, Sarthe, France.*

If an arbitrary straight line d, passing through any point P of the plane of a triangle ABC, meets the straight lines BC, CA and AB in points A_1, B_1 and C_1, and the points obtained in prolonging the segments A_1P, B_1P, and C_1P by three times their length are A_1', B_1', and C_1', then the mid-points of AA_1', BB_1' and CC_1', A_2, B_2, and C_2, respectively, are the vertices of a triangle, the area of which is equal to that of triangle ABC.

II. *Solution by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.* We may express the relations between the points by vectorial equalities and arrive at the desired result by vectorial multiplication. We first note that the point P is not necessarily on d. According to the notations as stated, we have

$$\vec{PA}_1' = -3\vec{PA}_1 \quad \text{and} \quad 2\vec{PA}_2 = \vec{PA} + \vec{PA}_1' = \vec{PA} - 3\vec{PA}_1$$

Now

$$2\vec{A_2B_2} = 2(\vec{PB}_2 - \vec{PA}_2) = (\vec{PB} - 3\vec{PB}_1 - \vec{PA} + 3\vec{PA}_1)$$

$$2\vec{A_2B_2} = \vec{AB} - 3\vec{A_1B_1}$$

and similarly

$$2\vec{A_2C_2} = \vec{AC} - 3\vec{A_1C_1}\ .$$

Multiplying the last two equalities member to member and denoting by $\overline{ABC}$ the area of the oriented triangle ABC we get

$$\begin{aligned} 4\overline{A_2B_2C_2} &= (\vec{AB} - 3\vec{A_1B_1}) \times (\vec{AC} - 3\vec{A_1C_1}) \\ &= \vec{AB} \times \vec{AC} - 3(\vec{AB} \times \vec{A_1C_1} + \vec{A_1B_1} \times \vec{AC}) + 9\vec{A_1B_1} \times \vec{A_1C_1} \end{aligned}$$

The last term being zero

$$\begin{aligned} 4\overline{A_2B_2C_2} &= \overline{ABC} - 3(\vec{AB} \times \vec{A_1A} + \vec{A_1A} \times \vec{AC}) \\ &= \overline{ABC} - 3(\vec{AB} - \vec{AC}) \times \vec{A_1A} \\ &= \overline{ABC} + 3\vec{BC} \times \vec{CA} = 4\overline{ABC} \quad \text{Q.E.D.} \end{aligned}$$

This problem may be generalized as follows : If $A_1B_1C_1$ is an inscribed triangle of ABC and if $\vec{PA}_1' = -n\vec{PA}_1$ (in the present case $n = 3$), $\vec{AA}_2 = m\vec{AA}_1'$ (in the present case $m = \frac{1}{2}$), then we have

$$\overline{A_2B_2C_2} = (1-m)(1-2m+mn)\overline{ABC} + m^2(1-m)^2\overline{A_1B_1C_1}$$

Also solved by Christopher Henrich (partially) and the proposer.

Contributed Solution to Proposal 383:
***Mathematics Magazine*, 33, (1959-1960), 228.**

Disecting a Square

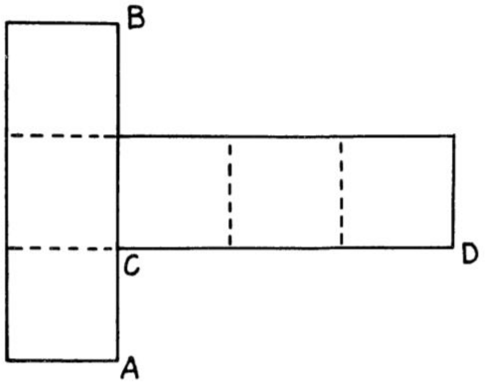
383. [September 1959] *Proposed by Raphael T. Coffman, Richland, Washington.*

Cut any square into not more than six pieces which can be reassembled to form a cube having its surface area equal to the area of the square. Bending of the pieces is permissible.

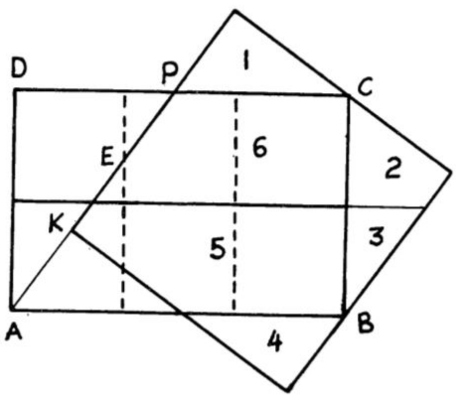
I. *Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.* Let a be the side of the square. The edge u of the cube is, by $6u^2 = a^2$, $u = a\sqrt{6}/6$. We develop the cube as shown in (1) and assemble the rectangles to form the rectangle $ABCD$ (2). Take $AP = \sqrt{AB \cdot AD} = \sqrt{3u \cdot 2u} = u\sqrt{6}$. Let BE be perpendicular to AB. Then from the similar triangles ABE and APD, having

$$BE : AB = AD : AP \quad \text{and} \quad BE = AB \cdot AD/AP = AP^2/AP = AP\ ,$$

we can draw the square shown in (2). Comparing (2) and (3) we see the equivalence of $ABCD$ and the square, the side of the latter being evidently a. The number of pieces is 6 and is less than 7.

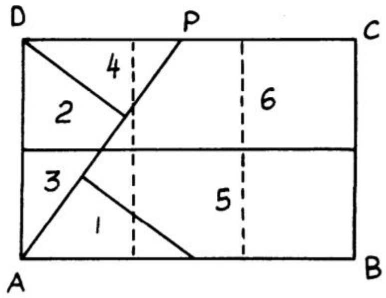


(1)

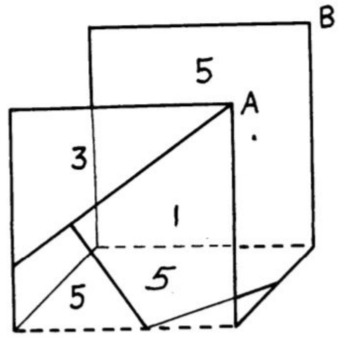


(2)

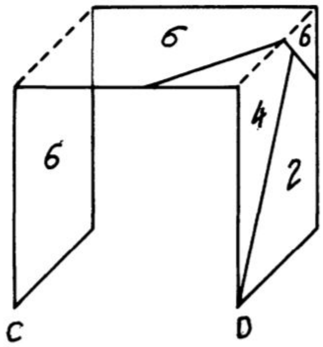
Now, how the cube is obtained is shown by the drawings (3a), (3b) and (4). The solution is therefore completed. If one needs to cut the square into pieces without the use of the rectangle $ABCD$ (2), note the dimensions of (5).



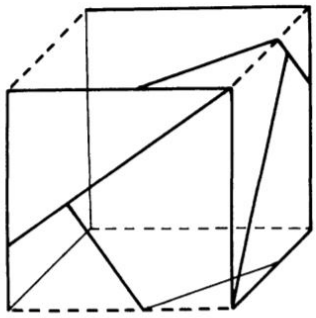
(3)



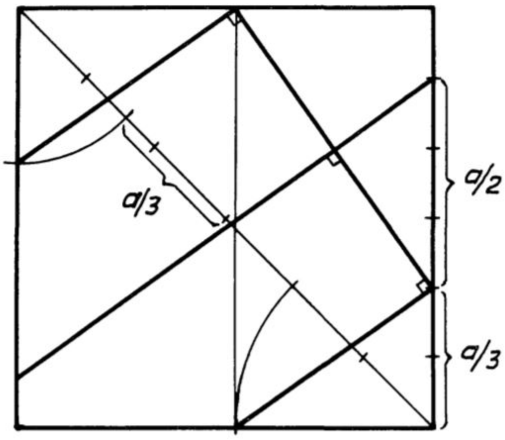
(3a)



(3b)



(4)



(5)

Contributed Solution to Proposal 387:
***Mathematics Magazine*, 33, (1959-1960), 233.**

An Induction Proof

387. [September 1959] *Proposed by D. S. Mitrinovitch, University of Belgrade, Yugoslavia.*

Prove the relation,

$$\left[\frac{\partial^n}{\partial t^n}\left(\frac{1}{1-t}e^{\frac{-xt}{1-t}}\right)\right]_{t=0} = e^x\frac{d^n}{dt^n}(x^n e^{-x})$$

n a natural number, by induction.

Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey. The equality is evidently true for $n = 0$. Supposing it be true for $n = p$, let us prove it for $n = p+1$. By hypothesis we have

$$\left[\frac{\partial^p}{\partial t^p}\left(\frac{1}{1-t}e^{-\frac{xt}{1-t}}\right)\right]_{t=0} = e^x\frac{d^p}{dx^p}(x^p e^{-x}) \tag{1}$$

The right hand side of (1) may be obtained from the Leibniz formula (D stands for d/dx)

$$e^x\frac{d^p}{dx^p}(x^p e^{-x}) = e^x\sum_{k=0}^{p}\binom{p}{k}D^k x^p\cdot D^{p-k}e^{-x} = e^x\sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}\frac{p!}{(p-k)!}x^{p-k}e^{-x}$$

$$= e^x\sum_{\lambda=0}^{p}(-1)^{\lambda}\binom{p}{\lambda}\frac{p!}{\lambda!}x^{\lambda}e^{-x}$$

As to the left hand side of (1), we have

$$\frac{\partial^p}{\partial t^p}\left(\frac{1}{1-t}e^{-\frac{xt}{1-t}}\right)=\frac{\partial^p}{\partial t^p}\left(\frac{1}{1-t}e^{x-\frac{x}{1-t}}\right) = e^x\frac{\partial^p}{\partial t^p}\left(\frac{1}{1-t}e^{-\frac{x}{1-t}}\right)$$

$$= e^x\frac{\partial^p}{\partial t^p}\frac{\partial}{\partial x}e^{-\frac{x}{1-t}} = e^x\frac{\partial}{\partial x}\frac{\partial^p}{\partial t^p}e^{-\frac{x}{1-t}}\ .$$

Replacing $u = e^{-\frac{x}{1-t}}$, (1) is reduced to

$$\left[e^x\frac{\partial}{\partial x}\frac{\partial^p}{\partial t^p}u\right]_{t=0} = e^u\sum_{\lambda=0}^{p}(-1)^\lambda\binom{p}{\lambda}\frac{p\,!}{\lambda\,!}x^\lambda e^{-x}\ . \tag{2}$$

Now, applying $\frac{\partial}{\partial t}u = x\frac{\partial^2}{\partial x^2}u$ to the left member of (2), we get

$$\frac{\partial}{\partial x}\frac{\partial^p}{\partial t^p}u=\frac{\partial}{\partial x}\frac{\partial^{p-1}}{\partial t^{p-1}}\left(x\frac{\partial^2}{\partial x^2}\right)u=\frac{\partial}{\partial x}\frac{\partial^{p-2}}{\partial t^{p-2}}\left(x\frac{\partial^2}{\partial x^2}\right)\left(x\frac{\partial^2}{\partial x^2}\right)u=\cdots=\frac{\partial}{\partial x}\left(x\frac{\partial^2}{\partial x^2}\right)^p u$$

and

$$\left[\frac{\partial}{\partial x}\left(x\frac{\partial^2}{\partial x^2}\right)^p u\right]_{t=0} = \frac{\partial}{\partial x}\left(x\frac{\partial^2}{\partial x^2}\right)^p e^{-x} = D(xD^2)^p e^{-x}\ .$$

The equality reduces therefore to (3)

$$D(xD^2)^p e^{-x} = \sum_{\lambda=0}^{p}(-1)^\lambda\binom{p}{\lambda}\frac{p\,!}{\lambda\,!}x^\lambda e^{-x}\ . \tag{3}$$

The proof of the statement is completed if we can prove (3) for p replaced by $p+1$. Now,

$$D(xD^2)^{p+1}e^{-x} = D(xD^2)(xD^2)^p e^{-x} = DxD[D(xD^2)^p e^{-x}]$$

$$= DxD\sum_{\lambda=0}^{p}(-1)^\lambda\binom{p}{\lambda}\frac{p\,!}{\lambda\,!}x^\lambda e^{-x} = DxD\sum_{\lambda=0}^{p}a_\lambda x^\lambda e^{-x}$$

$$= (a_1-a_0)+\sum_{\lambda=1}^{p-1}[(\lambda+1)^2 a_{\lambda+1}-(2\lambda+1)_{a_\lambda+a_{\lambda-1}}]x^\lambda e^{-x}+$$

$$[-(2p+1)a_p+a_{p-1}]x^p e^{-x}+a_p x^{p+1}e^{-x}$$

$$= -(p+1)\,!+\sum_{\lambda=1}^{p-1}b_\lambda x^\lambda e^{-x}+(-1)^{p-1}(p+1)^2x^pe^{-x}+(-1)^px^{p+1}e^{-x} \tag{4}$$

where the first and the last two coefficients are obtained through a_λ, and

$$b_\lambda = (\lambda+1)^2 a_{\lambda+1} - (2\lambda+1)a_\lambda + a_{\lambda-1} \qquad (1 \leqq \lambda \leqq p-1)$$

$$= (\lambda+1)^2(-1)^{\lambda+1}\binom{p}{\lambda+1}\frac{p!}{(\lambda+1)!} - (2\lambda+1)(-1)^\lambda\binom{p}{\lambda}\frac{p!}{\lambda!} + (-1)^{\lambda-1}\binom{p}{\lambda-1}\frac{p!}{(\lambda-1)!}$$

$$= (-1)^{\lambda-1}\frac{p!}{\lambda!(p-\lambda+1)!}\frac{p!}{\lambda!}[(p-\lambda)(p-\lambda+1) + (2\lambda+1)(p-\lambda+1) + \lambda^2]$$

$$= (-1)^{\lambda-1}\frac{p!}{\lambda!(p-\lambda+1)!}\frac{p!}{\lambda!}\cdot(p+1)^2 = (-1)^{\lambda-1}\frac{(p+1)!}{\lambda!(p-\lambda+1)!}\frac{(p+1)!}{\lambda!}$$

$$= (-1)^{\lambda-1}\binom{p+1}{\lambda}\frac{(p+1)!}{\lambda!}$$

It is easy to see that the first and the last two coefficients of (4) are b_0, b_p and b_{p+1}, and hence

$$D(xD^2)^{p-1}e^{-x} = \sum_{\lambda=0}^{p+1}(-1)^{\lambda-1}\binom{p+1}{\lambda}\frac{(p+1)!}{\lambda!}x^\lambda e^{-x}.$$

Thus the equality being proved for $n = p+1$, it will be true for all integral values of n and the proof is completed.

Contributed Solution to Proposal 400:
Mathematics Magazine, 34, (1960-1961), 53.

A Triangle Construction

400. [January 1960] *Proposed by W. B. Carver, Cornell University.*

Given a point, a circle, and any curve in a plane. Construct an equilateral triangle having a vertex on each of them.

II. *Solution by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Let A, (b), (c) be the given point, circle and curve. If, then, ABC is the solution, the vertex C on (c) is obtained from the vertex B on (b) by a rotation about A with the angles 60 and -60 degrees. Hence the unknown vertices on (c) are obtained by intersecting (c) with new positions of (b) after such rotations. The constructions of the triangles are then immediate.

Also solved by Harry M. Gehman, University of Buffalo; Rostyslaw J. Lewyckyj, University of Toronto; Harvey Walden (partial solution); and the proposer.

Contributed Solution to Proposal 401:
***Mathematics Magazine*, 34, (1960-1961), 55.**

A Fibonacci Series

401. [January 1960] *Proposed by John M. Howell, Los Angeles City College.*

Given a sequence of numbers related by $F(n) = aF(n-1) + bF(n-2)$, $F(0) = c$ and $F(1) = d$, where $n = 0, 1, 2, \cdots$ and a, b, c, and d are any real numbers. Find a general form for $F(n)$.

II. *Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

We have successively

$$F(0) = c \qquad\qquad = c$$

$$F(1) = d \qquad\qquad = d$$

$$F(2) = aF(1) + bF(0) \qquad\qquad = \binom{1}{0}ad + \binom{0}{0}bc$$

$$F(3) = aF(2) + bF(1)$$

$$= (a^2 + b)F(1) + abF(0) \qquad = \Big[\binom{2}{0}a^2 + \binom{1}{1}b\Big]d + \Big[\binom{1}{0}ab \;\; c$$

$$F(4) = aF(3) + bF(2)$$

$$= (a^3 + 2ab)F(1) + (a^2b + b^2)F(0) = \Big[\binom{3}{0}a^3 + \binom{2}{1}ab\Big]d + \Big[\binom{2}{0}a^2b + \binom{1}{1}b^2\Big]c$$

$$\cdots \qquad\qquad \cdots$$

and in general

$$F(n) = \Big[\binom{n-1}{0}a^{n-1} + \binom{n-2}{1}a^{n-3}b + \binom{n-3}{2}a^{n-5}b^2 + \cdots\Big]d$$

$$+\Big[\binom{n-2}{0}a^{n-2} + \binom{n-3}{1}a^{n-4}b + \binom{n-4}{2}a^{n-6}b^2 + \cdots\Big]bc$$

which may be proved by induction.

III. *Alternate solution by Huseyin Demir.*

Writing the relation for $n = 2, \cdots, n$ we have a system of equations in the unknowns $F(2), \cdots, F(n)$:

$$\begin{aligned}
F(n) - aF(n-1) - bF(n-2) &= 0\\
F(n-1) - aF(n-2) - bF(n-3) &= 0\\
&\cdots\\
F(4) - aF(3) - bF(2) &= 0\\
F(3) - aF(2) &= bF(1)\\
F(2) &= aF(1) - bF(0)
\end{aligned}$$

The determinant of the system being 1 we have

$$F(n) = \begin{vmatrix} 0 & -a & -b & 0 & \cdots & 0 \\ 0 & 1 & -a & -b & 0 & 0 \\ & 0 & 1 & -a & -b & \\ & & \cdots & & & \\ 0 & 0 & & 1 & -a & -b \\ bd & 0 & & 0 & 1 & -a \\ ad+bc & 0 & \cdots & & 0 & 1 \end{vmatrix}_{n-1}$$

where the index denotes the order of the determinant.

Expanding it with respect to the first column and arranging, we have the final result

$$F(n) = (bc+ad)\begin{vmatrix} a & b & 0 & \cdots & 0 \\ -b & a & b & & \vdots \\ 0 & -b & a & b & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ & & -b & a & b \\ 0 & \cdots & 0 & -b & a \end{vmatrix}_{n-2} + bd\begin{vmatrix} a & b & 0 & \cdots & 0 \\ -b & a & b & & \vdots \\ 0 & -b & a & b & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ & & -b & a & b \\ 0 & \cdots & 0 & -b & a \end{vmatrix}_{n-3}$$

IV. *Alternate solution by Huseyin Demir.*

Writing the given relation

$$F(n) = aF(n-1) + bF(n-2)$$

in the form

$$\frac{F(n)}{F(n-1)} = a + \frac{b}{F(n-1)/F(n-2)}$$

and letting $u_n = F(n)/F(n-1)$ we have successively

$$u_n = a + b/u_{n-1}$$

$$= a + \frac{b}{a + b/u_{n-2}}$$

$$\cdots$$

$$u_n = a + \cfrac{b}{a + \cfrac{b}{a + \ddots + \cfrac{b}{a + d/c}}}$$

Multiplying member to member the relations

$$F(n) = u_n F(n-1)$$

$$\cdot \quad \cdot \quad \cdot$$

$$F(2) = u_2 F(1)$$

$$F(1) = u_1 F(0)$$

we have for the general form for $F(n)$:

$$F(n) = u_1 u_2 \cdots u_n \cdot c$$

Also solved by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; R. G. Buschman, University of Oregon; F. D. Parker, University of Alaska; Charles F. Pinzka, University of Cincinnati; Chih-yi Wang, University of Minnesota; and the proposer.

Contributed Solution to Proposal 412:
***Mathematics Magazine*, 34, (1961), 175.**

Projective Correspondence

412. [May 1960] *Proposed by D. Moody Bailey, Princeton, West Virginia.*

P is any point on the circumcircle of triangle ABC. Rays from B and C through P meet CA and AB at points E and F respectively. Considering the segments involved as directed quantities, show that

$$\frac{b^2}{a^2}\cdot\frac{BF}{FA}+\frac{c^2}{a^2}\cdot\frac{CE}{EA}=-1\ ,$$

where a, b, and c are the sides opposite the vertices A, B, and C of triangle ABC.

Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.

Having a projective correspondence between the points E and F, we have, letting $e = CE/EA$, $f = BF/FA$, the bilinear relation

$$A\cdot ef+B\cdot e+C\cdot f+D=0$$

where A, B, C, D are constants. To find the values of these coefficients we let P coincide with the points A, B, C successively. If $P = A$, e and f are infinite and $A = 0$. If $P = B$, then BE is an exsymedian; and we have $e = -a^2/c^2$, $f = 0$ and hence

$$-\frac{B\cdot a^2}{c^2}+D=0 \quad \text{or} \quad B=\frac{c^2}{a^2}D$$

and similarly

$$-\frac{C\cdot a^2}{b^2}+D=0 \quad \text{or} \quad C=\frac{b^2}{a^2}D\ .$$

Substitution gives the required result.

Also solved by Josef Andersson, Vaxholm, Sweden; Leon Bankoff, Los Angeles, California; A. F. Hordam, University of New England, Armidale, NSW, Australia; and the proposer.

Contributed Solution to Proposal 427:
Mathematics Magazine, 34, (1961), 303.

A Cevian Relation

427. [November 1960] *Proposed by D. Moody Bailey, Princeton, West Virginia.*

P is any point in the plane of a triangle ABC through which cevians from B and C are drawn meeting sides CA and AB at points E and F reapectively. M is the midpoint of BC and line MP meets CA at N and AB at O. EF extended meets BC at G and a line through B parallel to AG meets CF at H. Show that HO is parallel to CA.

Solution by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.

Let the points A, B, C, M and F be fixed and the geometrically interrelated points P, O, N, G, E, H be variable. Then from

$$O \overset{M}{\doublebarwedge} P \overset{B}{\doublebarwedge} E \overset{F}{\doublebarwedge} G \doublebarwedge AG \doublebarwedge BH \doublebarwedge H$$

we have $O \barwedge H$ of which F being the self corresponding element we deduce the perspectivity $O \doublebarwedge H$. Hence OH passes through a fixed point L. When O is at infinity on AB, H is also at infinity on CF, and hence L is at infinity. OH keeps then a fixed direction. But when $O \equiv B$, having $OH \equiv BH//AB$ the proof follows.

Also solved by the proposer.

Contributed Solution to Proposal 428:
Mathematics Magazine, 34, (1961), 303.

Permuted Digits

428. [November 1960] *Proposed by Murray S. Klamkin, AVCO, Wilmington, Massachusetts.*

The number $N = 142{,}857$ has the property that $2N$, $3N$, $4N$, $5N$, and $6N$ are all permutations of N. Does there exist a number M such that $2M$, $3M$, $4M$, $5M$, $6M$, and $7M$ are all permutations of M?

I. *Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Since we get all permutations of M by $1M$, $2M$, $\cdots$, $7M$ the number M, if it exists, is a seven-digit number.

Let $M = abcdefg = Gg$ where $G = abcdef$ and let $1 \leqq p \leqq 7$ such that $p \cdot Gg = gG$. Then

$$p(10G+g) = 10^6 g + G$$

or

$$G = \frac{(10^6 - p)g}{(10p-1)} = N_p \cdot \frac{g}{Dp} .$$

Now

p	N_p	D_p	N_p/D_p	$(N_p/3)/D_p$
1	999,999	9	111,111	.
2	999,998	19	Irreducible	.
3	999,997	29	Irreducible	.
4	999,996	39 = 3.13	.	Irreducible
5	999,995	49 = 7.7	Irreducible	.
6	999,994	59	Irreducible	.
7	999,993	69 = 3.23	.	Irreducible

Since the coefficient N_p/D_p of g is not an integer except when $p = 1$, there is no solution for G other than $\overline{ggg,ggg}$. But $M = Gg = \overline{ggggggg}$ cannot be a solution.

Hence there is no solution to the problem.

II. *Comment by Dermott A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts.*

The number $M = 5882352941176470$ has the property that kM is a permutation of M for $k = 2, 3, \cdots, 16$. The number

$$L = 3448275862068965517241379310$$

has the property that kL is a permutation of L for $k = 2, 3, \cdots, 28$. (M consists of the digits in one cycle of the decimal expansion of $1/17$, and is 16 digits long, while L was similarly derived from $1/29$. I believe that it is correct that when p is prime and $1/p = Q$ has cycle length $p-1$, then kQ will be a permutation of Q for $k = 2, 3, \cdots, p-1$.)

Contributed Solution to Proposal 432:
Mathematics Magazine, 34, (1961), 365.

Cevian Lines

432. [January 1961] *Proposed by Lee Tih-Ming, Taipei, Taiwan.*

A point O interior to triangle ABC is joined to the vertices. From O perpendiculars OX, OY, OZ are dropped to the sides BC, CA, AB, respectively. AO and YZ intersect in D, BO and ZX in E, and CO and XY in F. Show that

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{ZD}{DY} \cdot \frac{YF}{FX} \cdot \frac{XE}{EZ} .$$

II. *Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*
The point O is not necessarily within the triangle. Letting

$$\alpha = \measuredangle BAO \qquad \beta = \measuredangle CBO \qquad \gamma = \measuredangle ACO$$
$$\alpha' = \measuredangle OAC \qquad \beta' = \measuredangle OBA \qquad \gamma' = \measuredangle OCB$$

we write from the triangles such as AZD and ADY, the relations

$$\frac{ZD}{\sin\alpha} = \frac{AZ}{\sin D}, \ \frac{DY}{\sin\alpha'} = \frac{YA}{\sin D} \quad \text{and} \quad \frac{ZD}{DY} = \frac{AZ}{YA}\cdot\frac{\sin\alpha}{\sin\alpha'}$$

and two others. Multiplying the three ratios member to member we obtain

$$\frac{ZD}{DY}\cdot\frac{YF}{FX}\cdot\frac{XE}{EZ} = \frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\left(\frac{\sin\alpha}{\sin\alpha'}\cdot\frac{\sin\beta}{\sin\beta'}\cdot\frac{\sin\gamma}{\sin\gamma'}\right).$$

But the expression in the parenthesis is 1, since AO, BO, CO are concurrent. Hence the equality is true for all points in the plane of ABC.

Also solved by Brother Alfred, St. Mary's College, California; Josef Andersson, Vaxholm, Sweden; C. W. Trigg, Los Angeles City College; Dale Woods, Oklahoma State University; and the proposer.

Contributed Solution to Proposal 435:
Mathematics Magazine, 34, (1961), 368.

Triangular Extrema

435. [January 1961] *Proposed by M. S. Klamkin, AVCO, Wilmington, Massachusetts.*

Determine the largest and the smallest equilateral triangles that can be inscribed in an ellipse.

Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.

Let $A_1A_2A_3$ be an equilateral triangle inscribed in the ellipse

(1) $$(x^2/a^2) + (y^2/b^2) = 1 \quad (E) \quad a > b$$

and let

(2) $$(x-u)^2 + (y-v)^2 - r^2 = 0 \quad (\Omega)$$

be the circle circumscribed to $A_1A_2A_3$. It cuts (E) at the fourth point $A_4(x_4, y_4)$.

Eliminating y between (1) and (2) we get an equation of fourth degree in x

$$c^4\cdot x^4 - 4a^2c^2u\cdot x^3 + \cdots = 0$$

of which the roots are x_1, x_2, x_3, x_4.

If we elimate x between (1) and (2), the corresponding equation will be

$$c^4\cdot y^4 + 4b^2c^2v\cdot y^3 + \cdots = 0$$

and the roots are y_1, y_2, y_3, y_4.

Since $A_1A_2A_3$ is an equilateral triangle, we have

$$x_1 + x_2 + x_3 = 3u$$
$$y_1 + y_2 + y_3 = 3v$$

and

$$x_4 = \sum x_i - 3u = \frac{4a^2u}{c^2} - 3u = \frac{(a^2+3b^2)u}{c^2}$$

(3)

$$y_4 = \sum y_i - 3v = -\frac{4b^2v}{c^2} - 3v = -\frac{(b^2+3a^2)v}{c^2} \ .$$

The coordinates (3) satisfying (1) we obtain the relation

(4) $$(u^2/\alpha^2) + (v^2/\beta^2) = 1$$

where

$$\alpha = \frac{ac^2}{a^2+3b^2}, \quad \beta = \frac{bc^2}{b^2+3a^2} \ .$$

Hence the centers of the circles (Ω) lie on the ellipse (4) of which $\alpha > \beta$.

Now since the largest and the smallest triangles correspond to the greatest and the smallest values of the radius r of the circle (Ω), we write

$$\begin{aligned} r^2 &= (x_4 - u)^2 + (y_4 - v)^2 \\ &= \frac{(a-\alpha)^2u^2}{\alpha^2} + \frac{(b+\beta)^2v^2}{\beta^2} \\ &= Au^2 + (b+\beta)^2 = Bv^2 + (a-\alpha)^2 \ . \end{aligned}$$

$dr/du = 0$ gives

$$u = 0 \quad \text{and} \quad r_1 = b + \beta \ .$$

Similarly $dr/dv = 0$ gives

$$r_2 = a - \alpha \ ,$$

and one may readily verify that $r_1 > r_2$.

Hence, the largest (smallest) equilateral triangles inscribed in the ellipse, are ones inscribed to the circles of center $u = 0$, $v = \pm\beta$ ($u = \pm\alpha$, $v = 0$) and radius $b + \beta$ ($a - \alpha$).

There are four solutions, two for the largest and two for the smallest triangles.

Constructions: The largest (smallest) triangles inscribed in an ellipse, have one of their vertices at the extremities of the minor (major) axis of the ellipse, the axis being the axis of symmetry of the triangle.

Also solved by Josef Andersson, Vaxholm, Sweden; J. W. Clawson, Collegeville, Pennsylvania (Two solutions); and J. W. Mellender, University

Contributed Solution to Proposal 445:
***Mathematics Magazine*, 35, (1962), 317.**

445. [March and November **1961**]. ***Comment by Huseyin Demir, Middle East Technical University, Ankara, Turkey.***

The proof needs a little modification. Read: Let the orthogonal projection of A, B and P on the line OM be A', B' and P'. In the remaining part of the proof all letters P are to be replaced by P' and in conclusion we have

$$\frac{MA^2-k}{MB^2-k}=\frac{2OM\cdot A'P'}{2OM\cdot B'P'}=\frac{A'P'}{B'P'}=\frac{PA}{PB}\,.$$

Contributed Solution to Proposal 476:
Mathematics Magazine, 35, (1962), 312.

Perimeter and Area Bisector

476. [March 1962]. *Proposed by Kaidy Tan, Fukien Normal College, China.*

Draw a straight line bisecting the perimeter and area of a given quadrilateral.

Solution by Huseyin Demir, Middle East Technical University, Ankara, Turkey.

We consider two cases according as the line intersects two adjacent or two opposite sides. Either case includes the case in which the line contained a vertex.

I. The line intersects two adjacent sides (fig. 1.)

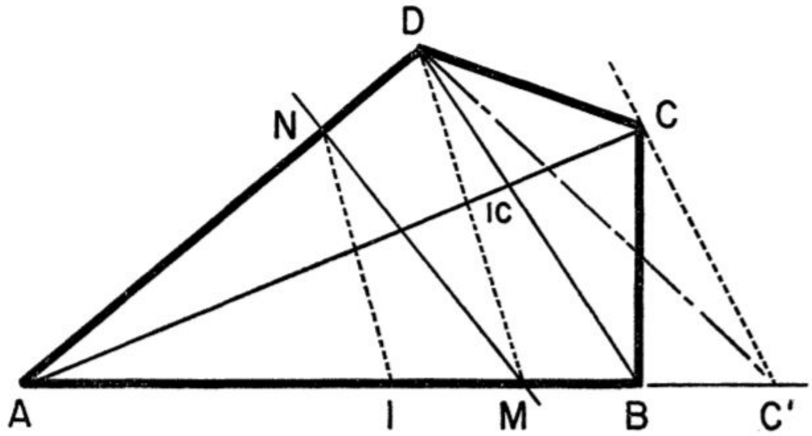


FIG. 1

(a) The line bisects the area:

Drawing $CC' \| BD$ we have area $ABCD = ABD + BCD = ABD + BC'D = AC'D$. Let I be the midpoint of AC', and M be a point on AB. Drawing $IN \| MD$ we have $AMN = AIN + IMN = AIN + IDN = AID = \frac{1}{2}AC'D = \frac{1}{2}ABCD$. Hence MN so constructed bisects the area. The constructions give:

Let $AM = m$, $AN = n$, then

$$AC'/AC = AB/AK = a/\alpha, \qquad AC' = a\cdot AC/\alpha = pa/\alpha.$$

$$AM/AD = AI/AN, \qquad mn = AI\cdot AD = \tfrac{1}{2}AC'\cdot d = pad/2\alpha.$$

$$2mn = pad/\alpha.$$

(b) The line bisects the perimeter:

$$m + n = AM + AN = \tfrac{1}{2}(a + b + c + d).$$

Therefore m, n determining the line MN are the roots of the quadratic equation:

$$2x^2 - (a + b + c + d)x + pad/\alpha = 0$$

For the existence of MN we have the conditions:

(1) $m \leq b$, $n \leq d$ or $a+b+c+d = 2m+2n \leq 2b+2d$ or $b+c \leq a+d$

(2) $pad/\alpha = 2mn \leq 2ad$ or $p \leq 2\alpha$, $\alpha + \gamma 2\alpha$, $\alpha \geq \gamma$.
(3) $\Delta = (a+b+c+d)^2 - 8pad/\alpha \geq 0$.

II. The line intersects two opposite sides.

(a) The line bisects the area:

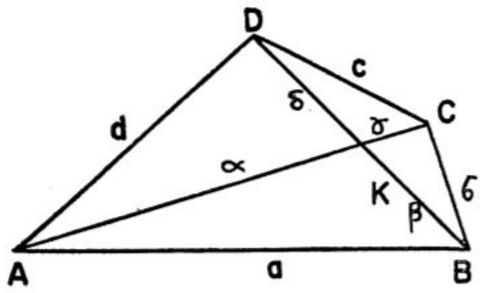


FIG. 2

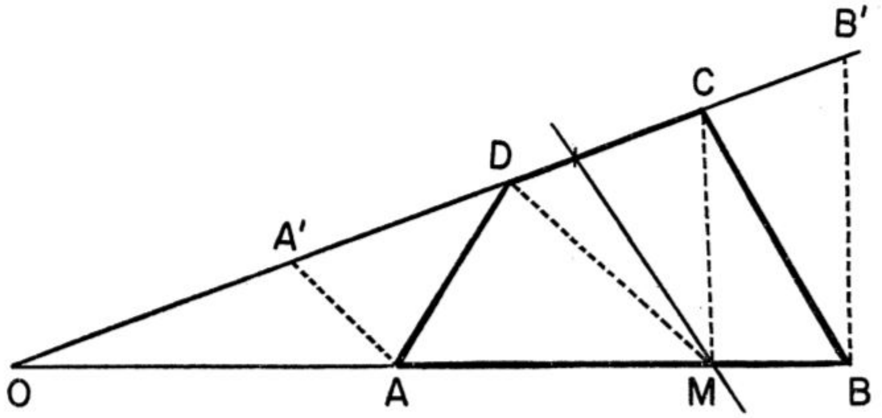


FIG. 3

Let M be any point on AB (fig. 3). Draw $BB' \| MC$, and $AA' \| MD$. Then: $ABCD = AMD + MCD + BCM = A'MD + MCD + B'CM = MDA' + MCD + MB'C = MB'A'$. If N is the midpoint of $A'B'$, the lkne MN will bisect $MB'A' = ABCD$. Let $OM = m$, $ON = n$, $OA = a'$, $OB = b'$, $OC = c'$, $OD = d'$, $OA' = a''$, $OB' = b''$. Then from the constructions:

$$m/c' = b'/b'', \qquad m/d' = a'/a''$$
$$2n = a'' + b'' = a'd'/m + b'c'/m = (a'd' + b'c')/m$$
$$2mn = a'd' + b'c'.$$

(b) The line bisects the perimeter:

$$MA + AD + DN = MB + BC + CN$$
$$(m - a') - d + (n - d') = (b' - m) + b + (c' - n)$$
$$2(m + n) = (a' + b' + c' + d') + (b - d.)$$

Therefore m, n determining the line MN are the roots of the quadratic equation

$$2x^2 - (a' + b' + c' + d' + b - d)x + (a'd' + b'c') = 0.$$

The existence of MN is given by $a' \leq m \leq b'$, $d' \leq m \leq c'$ which yield $a+c+b \geq d$ and $a+c+d \geq b$ which are always true.

Contributed Solution to Proposal 646:
***Mathematics Magazine*, 40, (1967), 226.**

The Complete Quadrilateral

646. [January, 1967] *Proposed by V. F. Ivanoff, San Carlos, California.*

Denoting the pairs of opposite vertices of a complete quadrilateral by A and A', B and B', C and C', respectively, prove that

$$\frac{AB \cdot AB'}{A'B \cdot A'B'} = \frac{AC \cdot AC'}{A'C \cdot A'C'} \,.$$

I. *Solution by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Writing the equality in the form

$$(AB/AC)(AB'/AC')(A'C'/A'B)(A'C/A'B') = 1$$

and replacing each fraction by its equivalent given by the sine law we have

$$(\sin C/\sin B)(\sin C'/\sin B')(\sin B/\sin C')(\sin B'/\sin C) = 1$$

which is an identity.

Contributed Solution to Proposal 653:
***Mathematics Magazine*, 42, (1969), 283.**

Exponential Derivative

653. [March, 1967] *Proposed by by Sam Newman, Atlantic City, New Jersey.*

What is dy/dx of

$$y = x^{\left. x^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}} \right\} n} \qquad ?$$

III. *Solution by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

The given function may be defined by the recurrence relation $y_n = x^{y_{n-1}}$, $y_1 = x^x$, $y_0 = x$, $y_{-1} = 1$. Taking logarithms and differentiating we obtain

$$\frac{y_n'}{y_n} = \frac{y_{n-1}'}{y_{n-1}}(y_{n-1}\ln x) + \frac{y_{n-1}}{x}$$

Writing the last equality from $n=1$ up to $n=n$ and multiplying each relating by a suitable factor and adding them up we get

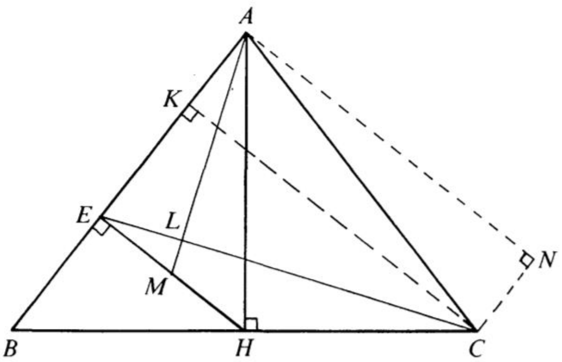
$$y_n' = \sum_{k=0}^{n} y_n y_{n-1} \cdots y_{n-k-1}(\ln x)^k/x.$$

Also solved by Pierre Bouchard, Université de Montréal, Canada; Nicholas C. Bystrom, St. Paul, Minnesota; Richard W. Feldman, Lycoming College, Pennsylvania; David Fettner, City College of New York; Reinaldo E. Giudici, University of Pittsburgh; Michael Goldberg, Washington, D.C.; Sandra A. Gossum, University of Tennessee; J. M. Howell, Los Angeles City College; Richard A. Jacobson, Houghton College, New York; Lew Kowarski, Morgan State College, Maryland; Fred Lambie, Lexington, Massachusetts; Douglas Lind, University of Virginia; Edwin A. Power, University College, London, England; and the proposer. A number of incorrect or undecipherable solutions were received.

Contributed Solution to Proposal 1199:
Mathematics Magazine, 58, (1985), 243.

Perpendicular Lines in an Isosceles Triangle **September 1984**

1199. In the isosceles triangle ABC, with $AB = AC$, let H be the foot of the altitude from A, let E be the foot of the perpendicular from H to AB, and let M be the midpoint of EH. Show that $AM \perp EC$. [*Aristomenis Siskakis, University of Illinois.*]



Solutions I *and* II: I. Let CK be the altitude from C. In the similar right triangles EHA and KBC, the corresponding medians AM and CE make equal angles with the hypotenuses HA and BC. Let these medians intersect at L. Then the quadrangle $HCAL$ is cyclic. Hence $\angle ALC = \angle AHC = 90°$.

II. We use harmonic pencils. Let CK be the altitude from C, and construct the rectangle $AKCN$. Since the segment HE, parallel to AN, is bisected by AM, we have $(AB, AH; AM, AN) = -1$. Similarly, since the segment BK, parallel to CN, is bisected by CE, we have $(CK, CB; CE, CN) = -1$. In the two harmonic pencils, three lines are perpendicular to corresponding lines. Hence the fourth lines, namely, AM and CE, are perpendicular.

HÜSEYIN DEMIR
Middle East Technical University
Ankara, Turkey

Also solved by sixty-two others (including the proposer and eight students), who submitted seventy-three solutions.

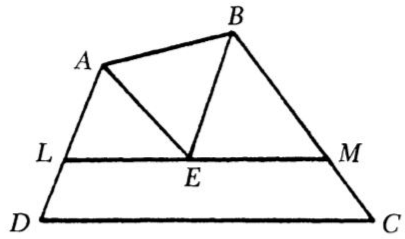
Joseph Konhauser located the problem in the *Monthly*, problem E1476, with three published solutions in v. 69 (1962), p. 233. Four other solvers of that problem forgot to mention the fact when submitting solutions to this problem. *P. J. Pedler* (Australia) and *J. H. Webb* (South Africa) found the problem in Loren C. Larson, *Problem Solving Through Problems*, p. 27, and *Geoffrey A. Kandall* found it in M. N. Aref & W. Wernick, *Problems and Solutions in Elementary Geometry*, p. 32, ex. 92. *O. Bottema* (The Netherlands) and *Webb* provided converses. (1) If ABC is any triangle, then $AM \perp EC$ if and only if $AB = AC$. (2) If E is any point on the line AB, then $AM \perp EC$ if and only if either $HE \perp AB$ or A is the midpoint of BE. (Other points are defined as in the problem statement.)

Contributed Solution to Proposal 1256:
***Mathematics Magazine*, 61, (1988), 54.**

Cyclic Quadrilateral **December 1986**

1256. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville.*

Let $ABCD$ be a cyclic quadrilateral, let the angle bisectors at A and B meet at E, and let the line through E parallel to side CD intersect AD at L and BC at M. Prove that $LA + MB = LM$.

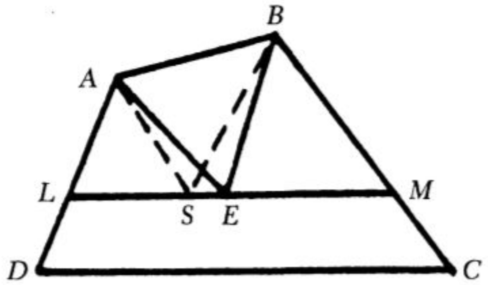


II. *Solution by H. Demir and C. Tezer, Middle East Technical University, Ankara, Turkey.*

Let $\angle DAB = 2\alpha$, $\angle ABC = 2\beta$, $\angle BCD = 2\gamma$, $\angle CDA = 2\delta$. Clearly, $\angle ELA = 2\delta$, $\angle BME = 2\gamma$, and $\alpha = \dfrac{\pi}{2} - \gamma$, $\beta = \dfrac{\pi}{2} - \delta$. We'll assume that $ABCD$ is convex and

$\alpha > \beta$.

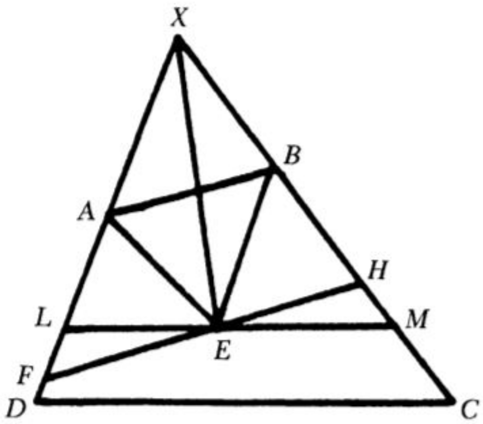
Choose a point S on LM on the same side of AD as M such that $|LS| = |LA|$ (see figure).



Obviously, $\angle ASL = \angle LAS = \beta$. Therefore, $ASEB$ is a cyclic quadrilateral. As $\angle LAS = \beta < \alpha = \angle LAE$, it follows that S is between L and E.

On the other hand, $\angle SBM = \angle SBE + \angle EBM = \angle SAE + \angle EBM = \angle LAE - \angle LAS + \beta = \alpha - \beta + \beta = \alpha = \angle BSM$. Consequently, MBS is an isosceles triangle and $|MS| = |MB|$. Therefore, $|LM| = |LS| + |SM| = |LA| + |MB|$.

III. *Solution by John P. Hoyt, Lancaster, Pennsylvania.*
Produce DA and CB to meet at X. Draw FH parallel to AB. Draw XE (see figure).



Since E is the intersection of two exterior angles of triangle XAB, XE is the bisector of $\angle AXB$. Triangles MLX and HFX are congruent because they have equal angles and a common angle bisector. The equal angles follow from the fact that the opposite angles of a cyclic quadrilateral are supplementary. Hence $ME = FE$, $HE = LE$, and $HM = LF$. Since FH is parallel to AB, and AE bisects $\angle DAB$, $\angle FAE = \angle AEF$. Thus, triangle FAE is isosceles, and $AF = FE$. Similarly, $BH = EH$.

The rest follows easily: $LA + MB = (AF - LF) + (BH + HM) = (AF + BH) + (HM - LF) = AF + BH = FE + EH = LM$.

Also solved by Frank Allen, Farid G. Bassiri (student), Andreas Bender (student, Switzerland), Nirdosh Bhatnagar, David Earnshaw (Canada), Howard Eves, Herta T. Freitag, Richard A. Gibbs, J. T. Groenman (Netherlands), Michael B. Handelsman, P. L. Hon (Hong Kong), King Jamison, Geoffrey A. Kandall, Tsz-Mie Ko (student), Mary S. Krimmel, L. Kuipers (Switzerland), Kee-wai Lau (Hong Kong), J. C. Linders (The Netherlands), David Morin (student, four solutions), Anna Michaelides Penk, Farhood Pouryoussefi (student, Iran), Harry D. Ruderman, Kiran Lall Shrestha (Nepal), J. M. Stark, M. Vowe (Switzerland), Harry Weingarten, and Brent Young (student).

Most of the solutions were based on trigonometric arguments (an impressive variety of trigonometric identities). A brillant, purely geometric, solution, due to Gregg Patruno (U.S.A.), appears in Murray Klamkin's International Mathematical Olympiads, 1978–1985, *New Mathematical Library, No. 31, MAA (solution to Problem 1, 1985).*