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Math 101 Calculus – Midterm Exam II - SOLUTIONS

Q-1) Does the function $f(x) = \dfrac{x^2-1}{x^4+1}$ have global minimum and maximum? If so, find at which points these extreme values are attained. If not, explain why?

Solution:

First observe that $\lim_{x\to\pm\infty} f(x) = 0$ and that the function takes both positive and negative values, so there is a global minimum and a global maximum.

$f'(x) = \dfrac{-2x^5+4x^3+2x}{(x^4+1)^2} = 0$ when $x = 0,\ \pm\sqrt{1+\sqrt{2}}$.

$f(\sqrt{1+\sqrt{2}}) = f(-\sqrt{1+\sqrt{2}}) > 0$ and $f(0) = -1$.

Hence the global maximum occurs at $x = \pm\sqrt{1+\sqrt{2}}$ and the global minimum occurs at $x = 0$.

Q-2) Find the area enclosed by the curve

$$f(x) = \frac{x^2}{\sqrt{x^3-1}}\,\frac{\sin^3\sqrt{x^3-1}}{\cos^5\sqrt{x^3-1}}$$

and the x-axis, between the points $x_1 = \left(1+\dfrac{\pi^2}{16}\right)^{1/3} \approx 1.17$ and $x_2 = \left(1+\dfrac{\pi^2}{9}\right)^{1/3} \approx 1.27$.

Solution:

Area$=\displaystyle\int_{x_1}^{x_2} f(x)\,dx = \int_{x_1}^{x_2} \frac{x^2}{\sqrt{x^3-1}}\ \tan\sqrt{x^3-1}\ \sec\sqrt{x^3-1}\,dx.$

Let $u = \tan\sqrt{x^3-1}$. Then $du = \dfrac{3}{2}\,\dfrac{x^2}{\sqrt{x^3-1}}\ \sec^2\sqrt{x^3-1}\,dx$, and $x = x_1 \Rightarrow u = \tan\pi/4 = 1$, $x = x_2 \Rightarrow u = \tan\pi/3 = \sqrt{3}$.

Now Area$=\displaystyle\frac{2}{3}\int_1^{\sqrt{3}} u^3\,du = \frac{2}{3}\left(\frac{1}{4}\,u^4\Big|_1^{\sqrt{3}}\right) = \frac{4}{3}.$

Q-3) Revolve the curve $f(x) = (\cos x)^{1/2}(1+\sin x)^{1/4}$ around the x-axis between the points $x_1 = 0$ and $x_2 = \pi/2$, and calculate the volume so obtained.

Solution:

$$\text{Volume}=\pi \int_0^{\pi/2} f(x)^2\, dx = \pi \int_0^{\pi/2} \cos x\, \sqrt{1+\sin x}\, dx.$$

Let $u = 1+\sin x$. Then $du = \cos x$ and $x = 0 \Rightarrow u = 1$, $x = \pi/2 \Rightarrow u = 2$.

$$\text{Then Volume}=\pi \int_1^2 u^{1/2}\, du = \frac{2\pi}{3} u^{3/2}\Big|_1^2 = \frac{2\pi}{3}\left(2^{3/2}-1\right) \approx 3.829.$$

Q-4) Revolve around y-axis the area bounded by the curve $f(x) = x\cos^2 x^3$ and the x-axis, between the points $x_1 = 0$ and $x_2 = \dfrac{\pi^{1/3}}{2}$. Find the volume so obtained.

Solution:

$$\text{Volume}=2\pi \int_{x_1}^{x_2} x f(x)\, dx = 2\pi \int_{x_1}^{x_2} x^2 \cos^2 x^3\, dx.$$

Let $u = x^3$. Then $du = 3x^2\, dx$, and $x = 0 \Rightarrow u = 0$, $x = (\pi^{1/3})/2 \Rightarrow u = \pi/8$.

$$\text{Volume}=\frac{2\pi}{3}\int_0^{\pi/8} \cos^2 u\, du = \frac{2\pi}{3}\int_0^{\pi/8} \frac{1+\cos 2u}{2}\, du = \frac{2\pi}{3}\left(\frac{1}{2}u + \frac{1}{4}\sin 2u\Big|_0^{\pi/8}\right)$$

$$= \frac{\pi^2}{24} + \frac{\pi}{6\sqrt{2}} \approx 0.781.$$

Q-5) Find the length of the curve given by the parametrization $x(t) = t^4$, $y(t) = 4t^7 - \dfrac{t}{7}$, between $t = 0$ and $t = 1$.

Solution:

$\dot{x} = 4t^3$, $\dot{y} = (4\cdot 7)t^6 - \frac{1}{7}$

$$\dot{x}^2 + \dot{y}^2 = 16t^6 + (4\cdot 7)^2 t^{12} - 8t^6 + \left(\tfrac{1}{7}\right)^2 = (4\cdot 7)^2 t^{12} + 8t^6 + \left(\tfrac{1}{7}\right)^2 = \left(28t^6 + \tfrac{1}{7}\right)^2.$$

$$\text{Hence Length} = \int_0^1 \sqrt{\dot{x}^2+\dot{y}^2}\, dt = \int_0^1 (28t^6 + \frac{1}{7})\, dt = \left(4t^7 + \frac{t}{7}\Big|_0^1\right) = \frac{29}{7}.$$

`Comments to sertoz@bilkent.edu.tr`