Math 101
March 21, 2009
Solutions

1) Write the derivatives of the following functions.
Do not simplify. (No partial credits.)

a) If $f(x) = \arcsin(\ln(x))$ then calculate $f'(x)$
Solution: By Chain Rule we have

$$f'(x) = \left(\frac{1}{\sqrt{1-(\ln(x))^2}}\right)\left(\frac{1}{x}\right)$$

b) If $g(x) = (\ln x)^{\tan(7x)}$ then calculate $g'(x)$
Solution: By logarithmic differentiation we have

$$g'(x) = \frac{d}{dx}\left(e^{\ln((\ln x)^{\tan(7x)})}\right) = \frac{d}{dx}\left(e^{\tan(7x)\ln((\ln x))}\right) =$$

$$= e^{\tan(7x)\ln((\ln x))}\left(7\sec^2(7x)\ln((\ln x)) + \tan(7x)\frac{1}{\ln(x)}\frac{1}{x}\right)$$

$$= (\ln x)^{\tan(7x)}\left(7\sec^2(7x)\ln((\ln x)) + \frac{\tan(7x)}{x\ln(x)}\right)$$

c) If $h(x) = \cos(e^{x^2+\tan^5(x^3)})$ then calculate $h'(x)$
Solution: By Chain Rule we have

$$h'(x) = -\sin(e^{x^2+\tan^5(x^3)})e^{x^2+\tan^5(x^3)}(2x + 15x^2\tan^4(x^3)\sec^2(x^3))$$

d) If $k(x) = x^{x^x}$ then calculate $k'(x)$
Solution: Note that by logarithmic differentiation

$$\frac{d}{dx}(x^x) = \frac{d}{dx}\left(e^{\ln(x^x)}\right) =$$

$$= \frac{d}{dx}\left(e^{x\ln(x)}\right) = e^{x\ln(x)}(\ln(x) + x\frac{1}{x}) = x^x(\ln(x)+1)$$

Hence by logarithmic differentiation

$$k'(x) = \frac{d}{dx}\left(e^{\ln(x^{x^x})}\right) = \frac{d}{dx}\left(e^{x^x\ln(x)}\right) =$$

$$= e^{x^x\ln(x)}\left(\frac{d}{dx}(x^x)\ln(x) + x^x(\frac{1}{x})\right) = x^{x^x}\left(x^x(\ln(x)+1)\ln(x) + x^{x-1}\right)$$

2) Find an equation of the tangent line and an equation of the normal line to the curve

$$xy^5 + x + y^2 = x^2y^3 + 5$$

at the point $(2, -1)$.

Solution: Apply $\frac{d}{dx}$ to both sides of the equation we get

$$y^5 + 5xy^4\frac{dy}{dx} + 1 + 2y\frac{dy}{dx} = 2xy^3 + 3x^2y^2\frac{dy}{dx}$$

Put $x = 2$ and $y = -1$ we get

$$-1 + 10\left(\left.\frac{dy}{dx}\right|_{(x,y)=(2,-1)}\right) + 1 - 2\left(\left.\frac{dy}{dx}\right|_{(x,y)=(2,-1)}\right) = -4 + 12\left(\left.\frac{dy}{dx}\right|_{(x,y)=(2,-1)}\right)$$

Hence

$$\left(\left.\frac{dy}{dx}\right|_{(x,y)=(2,-1)}\right) = 1$$

. Therefore the slope the tangent line to the curve $xy^5 + x + y^2 = x^2y^3 + 5$ at the point $(2, -1)$ is 1 and the slope the normal line to the curve $xy^5 + x + y^2 = x^2y^3 + 5$ at the point $(2, -1)$ is -1. Hence an equation of the tangent line to the curve $xy^5 + x + y^2 = x^2y^3 + 5$ at the point $(2, -1)$ is

$$y + 1 = x - 2$$

and an equation of the normal line to the curve $xy^5 + x + y^2 = x^2y^3 + 5$ at the point $(2, -1)$ is

$$y + 1 = -(x - 2)$$

3) A curve is given by the parametric equation:

$$x = t^3 + t + 2 \quad \text{and} \quad y = 2\cos(\frac{\pi}{2}t) + \ln(t)$$

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $t = 1$.

Solution: We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\pi\sin(\frac{\pi}{2}t) + \frac{1}{t}}{3t^2 + 1}$$

Therefore

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{-\pi + 1}{4}$$

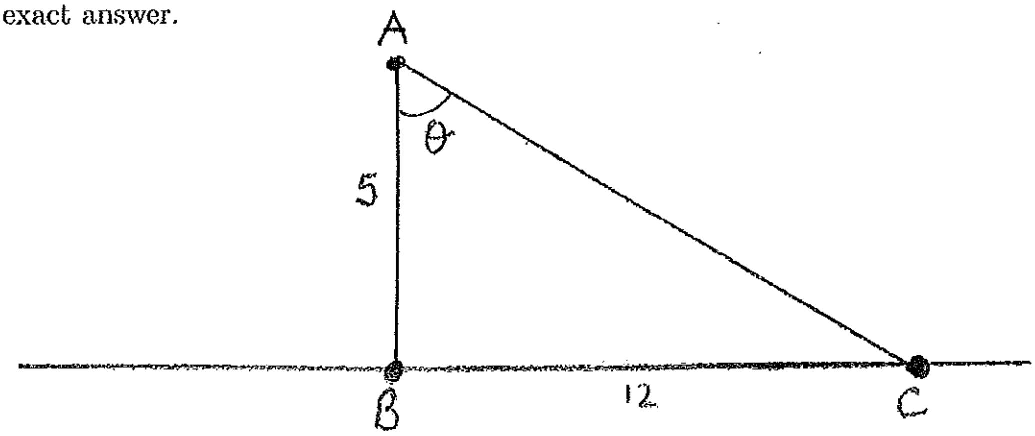
We have

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{\left(-\frac{\pi^2}{2}\cos(\frac{\pi}{2}t) - \frac{1}{t^2}\right)(3t^2+1) - \left(-\pi\sin(\frac{\pi}{2}t) + \frac{1}{t}\right)(6t)}{(3t^2+1)^2}}{3t^2 + 1}$$

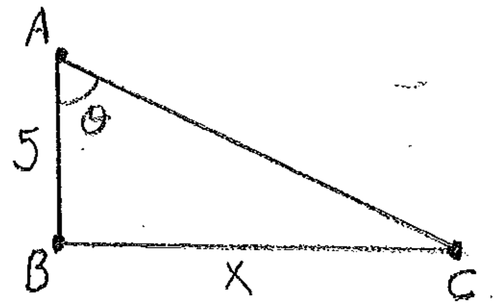
Therefore

$$\left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{6\pi - 10}{64}$$

4) There is a camera at the point A, 5 meters away from the race course which is a straight line. (See the figure below.) The camera at A is tracking an athlete as he runs along the course from C to B. Find the athlete's speed when he is at C, 12 meters away from B and the angle θ is decreasing at the rate of 0.7 radians/sec. Find an exact answer.



Solution: We first give names to lengths of the sides of the triangle above.



Then we have $\tan(\theta) = \frac{x}{5}$. Applying $\frac{d}{dt}$ to both sides we get $\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}$ When $x = 12$ we have $\sec(\theta) = \frac{13}{5}$. Therefore when $x = 12$ and $\frac{d\theta}{dt} = -\frac{7}{10}$ we have

$$\left(\frac{13}{5}\right)^2\left(-\frac{7}{10}\right) = \frac{1}{5}\frac{dx}{dt}$$

Therefore $\frac{dx}{dt}$ is $-\frac{1183}{50}$ when the athlete is at C, 12 meters away from B and the angle θ is decreasing at the rate of 0.7 radians/sec. This means his speed is $\frac{1183}{50}$ meters/sec at that instance.