1. Find $\left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(2,1)}$ if y is a differentiable function of x satisfying the equation $1+\ln(2x-3y) = x^2-3y^2$.

d/dx

$$\frac{1}{2x-3y}\cdot(2-3y') = 2x-6yy'$$

$\Downarrow$ $(x,y)=(2,1)$

$$2-3y' = 4-6y' \Rightarrow y' = \frac{2}{3} \text{ at } (x,y)=(2,1)$$

d/dx

$$-\frac{1}{(2x-3y)^2}\cdot(2-3y')^2 + \frac{1}{2x-3y}\cdot(-3y'') = 2-6y'\cdot y'-6yy''$$

$\Downarrow$ $(x,y)=(2,1)$, $y'=\frac{2}{3}$

$$-3y'' = 2-6\cdot\left(\frac{2}{3}\right)^2-6y''$$

$\Downarrow$

$$y'' = -\frac{2}{9} \text{ at } (x,y)=(2,1)$$

2. In each of the following, indicate whether the given statement is TRUE or FALSE by marking the corresponding ☐ with a ✗, and then **explain** why it is true or false.

a. $\displaystyle\int \sin^3 x\,dx = \frac{\sin^4 x}{4} + C$ ☐ TRUE ☒ FALSE

$\left(\frac{\sin^4 x}{4}\right)' = \sin^3 x \cos x \neq \sin^3 x$ at $x = \frac{\pi}{2}$

b. $\displaystyle\int \sin^3 x\,dx = \frac{\sin^4 x}{4\cos x} + C$ ☐ TRUE ☒ FALSE

$\left(\frac{\sin^4 x}{4\cos x}\right)' = \sin^3 x + \frac{\sin^5 x}{4\cos^2 x} \neq \sin^3 x$ at $x = \frac{\pi}{4}$

c. $\displaystyle\int \sin^3 x\,dx = \frac{1}{6}\cos x\cos 2x - \frac{5}{6}\cos x + C$ ☒ TRUE ☐ FALSE

$\left(\frac{1}{6}\cos x\cos 2x - \frac{5}{6}\cos x\right)' = \left(\frac{1}{6}\cos x\cdot(2\cos^2 x - 1) - \frac{5}{6}\cos x\right)'$

$= \left(\frac{1}{3}\cos^3 x - \cos x\right)' = -\cos^2 x\sin x + \sin x = \sin^3 x$ for all x

d. $\displaystyle\int \frac{dx}{x^2+1} = \frac{\ln(x^2+1)}{2x} + C$ ☐ TRUE ☒ FALSE

$\left(\frac{\ln(x^2+1)}{2x}\right)' = \frac{1}{x^2+1} - \frac{\ln(x^2+1)}{2x^2} \neq \frac{1}{x^2+1}$ at $x = 1$

e. $\displaystyle\int \frac{dx}{x^2+1} = \arcsin\left(\frac{x}{\sqrt{x^2+1}}\right) + C$ ☒ TRUE ☐ FALSE

$\left(\arcsin\left(\frac{x}{\sqrt{x^2+1}}\right)\right)' = \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{x^2+1}}\right)^2}} \cdot \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1} = \frac{1}{x^2+1}$ for all x

3. Some students believe that Bilkent Math 101 exams get more difficult as time passes. This is in fact true. The difficulty level $H(t)$ of these exams satisfies the differential equation

$$\frac{dH}{dt} = H^a$$

with the initial condition $H(0) = 1$, where t is time measured from Fall 1986 in academic years and a is a constant whose value is a secret.

There is a quatrain in Nostradamus's *Les Propheties* which can be interpreted to be about Bilkent Math 101 exams.

a. According to one interpretation, the exams will become infinitely difficult in Fall 2021. Accepting this interpretation, find the difficulty level of the exams in Fall 2016.

$a=1 \Rightarrow \frac{dH}{H} = dt \Rightarrow \ln|H| = t + C \Rightarrow \ln|H| = t \Rightarrow |H| = e^t \Rightarrow H = e^t$

$a \neq 1 \Rightarrow H^{-a}\,dH = dt \Rightarrow \frac{H^{-a+1}}{-a+1} = t + C \Rightarrow \frac{H^{-a+1}}{-a+1} = t + \frac{1}{-a+1} \Rightarrow H = ((1-a)t+1)^{1/1-a}$

$\lim\limits_{t\to 35^-} H = \infty \Rightarrow (1-a)\cdot 35 + 1 = 0$ and $\frac{1}{1-a} < 0 \Rightarrow a = \frac{36}{35}$

$\Rightarrow H = \left(1 - \frac{t}{35}\right)^{-35} \Rightarrow H(30) = 7^{35}$

The exams of 2016 are 7^{35} times as difficult as those of 1986.

b. According to another interpretation, the exams will be twice as difficult in Fall 2021 as they were in Fall 1986. Show that a must satisfy $-8 < a < -7$ if this is the case.

$e^{35} > e > 2 \Rightarrow a \neq 1. \Rightarrow ((1-a)\cdot 35+1)^{1/1-a} = 2$

$(36-35a)^{1/1-a} = 2 \iff 36 - 35a = 2^{1-a}$ and $a \neq 1$.

Let $f(a) = 36 - 35a - 2^{1-a}$.

$f'(a) = -35 + \ln 2 \cdot 2^{1-a} = 0 \Rightarrow a = \log_2\left(\frac{2\ln 2}{35}\right)$

$\Rightarrow$ By Rolle's Theorem, f can have at most one more zero beside $a=1$

$\left.\begin{array}{l} f(-8) = -196 < 0 \\ f(-7) = 25 > 0 \end{array}\right\} \Rightarrow f$ has a zero between -8 and -7 by IVT.

Therefore, $((1-a)\cdot 35+1)^{1/1-a} = 2$ has exactly one root and it lies in $(-8,-7)$.

4. Evaluate the limit $\lim_{x\to 0} \dfrac{e^{x-\frac{1}{2}x^2} - \arctan x - 1}{x^4}$.

$$\lim_{x\to 0} \frac{e^{x-\frac{1}{2}x^2} - \arctan x - 1}{x^4} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{(1-x)\,e^{x-\frac{1}{2}x^2} - \frac{1}{x^2+1}}{4x^3}$$

$$= \frac{1}{4} \lim_{x\to 0} \frac{(1-x+x^2-x^3)\,e^{x-\frac{1}{2}x^2} - 1}{x^3+x^5}$$

$$\overset{\text{L'H}}{=} \frac{1}{4} \lim_{x\to 0} \frac{(-1+2x-3x^2)\,e^{x-\frac{1}{2}x^2} + (1-x+x^2-x^3)(1-x)\,e^{x-\frac{1}{2}x^2}}{3x^2+5x^4}$$

$$= \frac{1}{4} \lim_{x\to 0} \frac{(-1+2x-3x^2+1-x+x^2-x^3-x+x^2-x^3+x^4)\,e^{x-\frac{1}{2}x^2}}{3x^2+5x^4}$$

$$= \frac{1}{4} \lim_{x\to 0} \frac{(-1-2x+x^2)\,e^{x-\frac{1}{2}x^2}}{3+5x^2}$$

$$= -\frac{1}{12}$$

5. Evaluate the following integrals.

a. $\displaystyle\int_0^1 x^5 \sqrt[4]{1-x^3}\,dx = -\frac{1}{3}\int_1^0 (1-u)\cdot u^{1/4}\,du = \frac{1}{3}\int_0^1 \left(u^{1/4} - u^{5/4}\right) du$

$u = 1 - x^3$

$du = -3x^2\,dx$

$\displaystyle = \frac{1}{3}\left[\frac{u^{5/4}}{5/4} - \frac{u^{9/4}}{9/4}\right]_0^1 = \frac{1}{3}\left(\frac{4}{5} - \frac{4}{9}\right) = \frac{16}{135}$

b. $\displaystyle\int \sin 2x \tan^2 x\,dx = \int 2\sin x\cos x \cdot \frac{\sin^2 x}{\cos^2 x}\,dx = 2\int \frac{1-\cos^2 x}{\cos x}\cdot \sin x\,dx$

$\displaystyle = -2\int\left(\frac{1}{u} - u\right) du = -2\ln|u| + u^2 + C' = \cos^2 x - 2\ln|\cos x| + C'$

$u = \cos x$

$du = -\sin x\,dx$