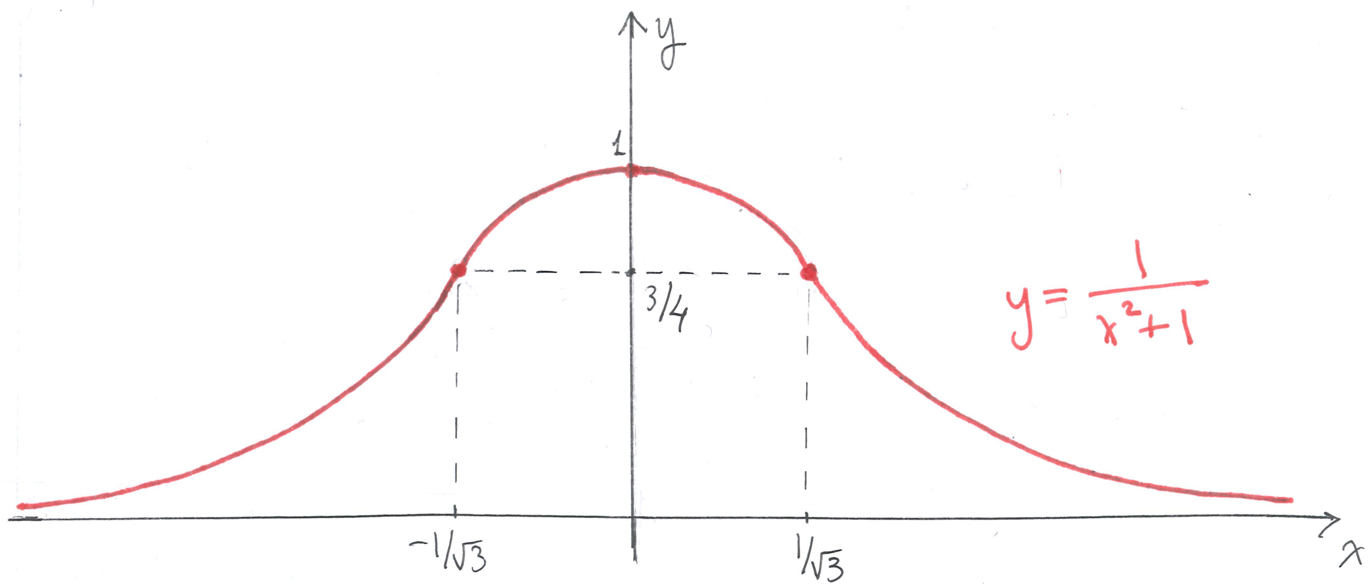
1a. Sketch the graph of $y = 1/(x^2+1)$ by computing y' and y'', and determining their signs; finding the critical points, the inflections points, the intercepts, and the asymptotes; and clearly labeling them in the picture.

$$y' = -\frac{2x}{(x^2+1)^2} = 0 \Rightarrow x = 0 \Rightarrow y = 1$$

$$y'' = -2 \cdot \frac{1\cdot(x^2+1)^2 - x \cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4} = \frac{2\cdot(3x^2-1)}{(x^2+1)^3} = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}} \Rightarrow y = \frac{3}{4}$$

x		$-1/\sqrt{3}$		0		$1/\sqrt{3}$	
y'	$+$		$+$	0	$-$		$-$
y''	$+$	0	$-$		$-$	0	$+$
		↑ inf pt		↑ loc max		↑ inf pt	

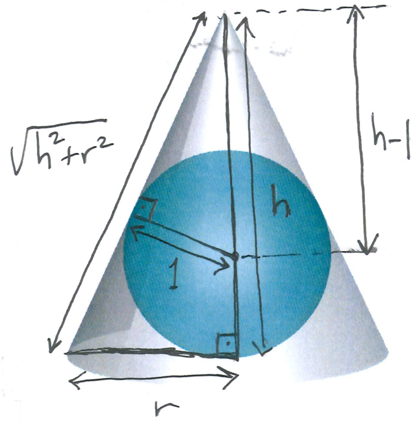
$$\lim_{\substack{x\to\infty \\ \text{or } -\infty}} y = 0$$



1b. Show that $\left|\frac{1}{a^2+1} - \frac{1}{b^2+1}\right| \le \frac{3\sqrt{3}}{8}|a-b|$ for all real numbers a, b.

Since $f(x) = 1/(x^2+1)$ is differentiable hence continuous everywhere, by MVT $f'(c) = \frac{f(a)-f(b)}{a-b}$ for some c between a and b. By Part a, critical points of f' are $x = \pm\frac{1}{\sqrt{3}}$, and $f'(\pm\frac{1}{\sqrt{3}}) = \mp\frac{3\sqrt{3}}{8}$. Moreover, $\lim_{\substack{x\to\infty \\ \text{or } -\infty}} f'(x) = 0$. Hence $|f'(x)| \le \frac{3\sqrt{3}}{8}$ for all x. In particular, $|f'(c)| \le \frac{3\sqrt{3}}{8}$ and therefore $\left|\frac{1}{a^2+1} - \frac{1}{b^2+1}\right| \le \frac{3\sqrt{3}}{8}\cdot|a-b|$.

2a. Find the dimensions of the cone with the smallest possible lateral surface area that surrounds a sphere of radius 1.



By similarity, $\dfrac{h-1}{1} = \dfrac{\sqrt{h^2+r^2}}{r} \Rightarrow r^2 = \dfrac{h}{h-2}$

A = Lateral Area $= \pi r \sqrt{r^2+h^2} = \pi r^2 (h-1) = \pi \cdot \dfrac{h(h-1)}{h-2}$

$$\boxed{\text{Minimize} \quad A = \pi \cdot \frac{h(h-1)}{h-2} \quad \text{for} \quad 2 < h < \infty}$$

$$\frac{1}{\pi}\frac{dA}{dh} = \frac{(2h-1)\cdot(h-2) - (h^2-h)\cdot 1}{(h-2)^2} = 0 \Rightarrow h^2 - 4h + 2 = 0$$

$\Downarrow$

$h = 2+\sqrt{2}$ or $\cancel{h = 2-\sqrt{2}}$ not in the interval

"Endpoints": $h = 2 \Rightarrow \lim\limits_{h\to 2^+} A = \infty$

$h = \infty \Rightarrow \lim\limits_{h\to\infty} A = \infty$

$\Rightarrow$ Smallest value occurs for $h = 2+\sqrt{2}$

$h = 2+\sqrt{2} \Rightarrow r = \sqrt{1+\sqrt{2}}$

The minimum lateral surface area occurs when $h = 2+\sqrt{2}$ and $r = \sqrt{1+\sqrt{2}}$.

2b. Having solved **Part 2a** correctly, after the exam you run into a friend who mistakenly minimized the total surface area $A = \pi r\sqrt{r^2+h^2} + \pi r^2$ instead of the lateral surface area $A = \pi r\sqrt{r^2+h^2}$.

Let h_y and r_y be respectively the height and the radius you found, and let h_f and r_f be the ones your friend did. Assuming that your friend also solved the corresponding problem correctly, which one of the following is true? No explanation is required.

☐ $h_y > h_f$ and $r_y > r_f$	☐ $h_y > h_f$ and $r_y = r_f$	☐ $h_y > h_f$ and $r_y < r_f$
☐ $h_y = h_f$ and $r_y > r_f$	☐ $h_y = h_f$ and $r_y = r_f$	☐ $h_y = h_f$ and $r_y < r_f$
☑ $h_y < h_f$ and $r_y > r_f$	☐ $h_y < h_f$ and $r_y = r_f$	☐ $h_y < h_f$ and $r_y < r_f$

3a. Determine $f(5)$ if f is a continuous function that satisfies $\int_0^{4x+\sin(\pi x)} f(t)\,dt = x^2$ for all x.

$$\frac{d}{dx}\int_0^{4x+\sin(\pi x)} f(t)\,dt = \frac{d}{dx}x^2$$

$$\Downarrow \leftarrow \boxed{\text{FTC 1}}$$

$$f(4x+\sin(\pi x))\cdot(4+\pi\cos(\pi x)) = 2x$$

$$\boxed{x=\frac{3}{2}} \rightarrow \Downarrow$$

$$f\left(4\cdot\frac{3}{2}+\sin\left(\frac{3\pi}{2}\right)\right)\cdot\left(4+\pi\cos\left(\frac{3\pi}{2}\right)\right)=2\cdot\frac{3}{2}$$

$$\Downarrow$$

$$f(5)=\frac{3}{4}$$

$\left[(4x+\sin\pi x)' = 4+\pi\cos\pi x \geq 4-\pi > 0 \Rightarrow 4x+\sin\pi x\right.$ is increasing on $(-\infty,\infty) \Rightarrow 4x+\sin(\pi x)=5$ only when $\left. x=\frac{3}{2}\right]$

3b. Determine $f(5)$ if f is a function that satisfies $\int_0^{f(x)} t^2\,dt = 4x+\sin(\pi x)$ for all x.

$$\left[\frac{1}{3}t^3\right]_0^{f(x)} = 4x+\sin(\pi x)$$

$$\Downarrow$$

$$\frac{1}{3}f(x)^3 = 4x+\sin(\pi x)$$

$$\Downarrow$$

$$f(x)=\left(12x+3\sin(\pi x)\right)^{1/3}$$

$$\boxed{x=5} \rightarrow \Downarrow$$

$$f(5)=\left(12\cdot 5+3\sin 5\pi\right)^{1/3}$$

$$\Downarrow$$

$$f(5)=60^{1/3}$$

4. Consider the region R between the graph of $y = x\sin(x^3)$ and the x-axis for $0 \le x \le \pi^{1/3}$.

a. Find the volume V of the solid generated by revolving R about the x-axis.

$$V = \pi \int_0^{\pi^{1/3}} \left(x\sin(x^3)\right)^2 dx$$

$$= \pi \int_0^{\pi^{1/3}} x^2 \sin^2(x^3)\, dx$$

$$= \pi \int_0^{\pi^{1/3}} x^2 \cdot \frac{1-\cos(2x^3)}{2}\, dx$$

$$= \frac{\pi}{2}\int_0^{\pi^{1/3}} x^2\, dx - \frac{\pi}{2}\int_0^{\pi^{1/3}} x^2 \cos(2x^3)\, dx$$

$$u = 2x^3$$
$$du = 6x^2\, dx$$

$$= \frac{\pi}{2}\cdot\left[\frac{1}{3}x^3\right]_0^{\pi^{1/3}} - \frac{\pi}{2}\int_0^{2\pi} \cos u \cdot \frac{1}{6}\, du$$

$$= \frac{\pi^2}{6} - \frac{\pi}{12}\left[\sin u\right]_0^{2\pi} = \frac{\pi^2}{6}$$

b. Find the volume W of the solid generated by revolving R about the y-axis.

$$W = 2\pi \cdot \int_0^{\pi^{1/3}} x \cdot x\sin(x^3)\, dx$$

$$= 2\pi \int_0^{\pi^{1/3}} x^2 \sin(x^3)\, dx$$

$$u = x^3$$
$$du = 3x^2\, dx$$

$$= 2\pi \int_0^{\pi} \sin u \cdot \frac{1}{3}\, du$$

$$= \frac{2\pi}{3}\left[-\cos u\right]_0^{\pi} = \frac{4\pi}{3}$$