1. Find $\left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(2\,,1/2)}$ if y is a differentiable function of x satisfying the equation:

$$\sin(\pi x y) = \sin(\pi x) + \cos(\pi y)$$

$$\Big\Downarrow \; d/dx$$

$$\cos(\pi x y)\cdot\pi\cdot(y + x y') = \cos(\pi x)\cdot\pi - \sin(\pi y)\cdot\pi\cdot y'$$

$$\Big\Downarrow \leftarrow \boxed{(x,y) = (2, \tfrac{1}{2})}$$

$$\cos(\pi\cdot 2\cdot\tfrac{1}{2})\cdot\pi\cdot(\tfrac{1}{2} + 2y') = \cos(\pi\cdot 2)\cdot\pi - \sin(\pi\cdot\tfrac{1}{2})\cdot\pi\cdot y'$$

$$\Big\Downarrow$$

$$-\pi\cdot(\tfrac{1}{2} + 2y') = \pi - \pi\cdot y'$$

$$\Big\Downarrow$$

$$y' = -\frac{3}{2} \quad \text{at } (x,y) = (2, \tfrac{1}{2})$$

d/dx

$$-\sin(\pi x y)\cdot\pi^2\cdot(y + x y')^2 + \cos(\pi x y)\cdot\pi\cdot(y' + y' + x y'')$$

$$= -\sin(\pi x)\cdot\pi^2 - \cos(\pi y)\cdot\pi^2\cdot(y')^2 - \sin(\pi y)\cdot\pi\cdot y''$$

$$\Big\Downarrow \leftarrow \boxed{(x,y) = (2, \tfrac{1}{2}),\; y' = -\tfrac{3}{2}}$$

$$-\sin(\pi\cdot 2\cdot\tfrac{1}{2})\cdot\pi^2\cdot\left(\tfrac{1}{2} + 2\cdot(-\tfrac{3}{2})\right)^2 + \cos(\pi\cdot 2\cdot\tfrac{1}{2})\cdot\pi\cdot\left(2\cdot(-\tfrac{3}{2}) + 2y''\right)$$

$$= -\sin(\pi\cdot 2)\cdot\pi^2 - \cos(\pi\cdot\tfrac{1}{2})\cdot\pi^2\cdot\left(-\tfrac{3}{2}\right)^2 - \sin(\pi\cdot\tfrac{1}{2})\cdot\pi\cdot y''$$

$$\Big\Downarrow$$

$$y'' = 3 \quad \text{at } (x,y) = (2, \tfrac{1}{2})$$

2. Evaluate the following.

a. $\displaystyle\int \frac{\tan x}{1+\sec 2x}\,dx = \int \frac{1}{1+\frac{1}{\cos 2x}}\cdot \tan x\,dx = \int \frac{\cos 2x}{1+\cos 2x}\cdot \tan x\,dx$

$\displaystyle = \int \frac{2\cos^2 x - 1}{2\cos^2 x}\cdot\frac{\sin x}{\cos x}\,dx = \int\left(\frac{1}{\cos x} - \frac{1}{2\cos^3 x}\right)\cdot \sin x\,dx$

$\displaystyle = \int\left(\frac{1}{u} - \frac{1}{2u^3}\right)\cdot(-du) = -\ln|u| - \frac{1}{4u^2} + C = -\ln|\cos x| - \frac{1}{4\cos^2 x} + C$

$\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x\,dx \end{array}}$

b. $\displaystyle\int_1^2 \ln(x^2+x)\,dx = \int_1^2 \ln((x+1)\cdot x)\,dx = \int_1^2 (\ln(x+1) + \ln x)\,dx$

$\displaystyle = \Big[(x+1)\ln(x+1) - (x+1) + x\ln x - x\Big]_1^2$

$= 3\ln 3 - 3 + 2\ln 2 - 2 - 2\ln 2 + 2 - 1\cdot\underbrace{\ln 1}_{0} + 1$

$= 3\ln 3 - 2$

c. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{in^2}{n^4+i^4} = \lim_{n\to\infty}\underbrace{\sum_{i=1}^{n}\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^4}\cdot\frac{1}{n}}_{\circledast} = \int_0^1 \frac{x}{1+x^4}\,dx = \frac{1}{2}\int_0^1 \frac{du}{1+u^2}$

$\boxed{\begin{array}{l} u = x^2 \\ du = 2x\,dx \end{array}}$

$\displaystyle = \frac{1}{2}\arctan u\Big]_0^1 = \frac{1}{2}\underbrace{\arctan 1}_{\frac{\pi}{4}} - \frac{1}{2}\underbrace{\arctan 0}_{0} = \frac{\pi}{8}$

$\circledast$ = a right Riemann sum for $f(x) = \dfrac{x}{1+x^4}$ on $[0,1]$

3. In each of the following, if the given statement is true for all functions f that are defined on $(-\infty, \infty)$, then mark the ☐ to the left of TRUE with a ✗; otherwise, mark the ☐ to the left of FALSE with a ✗ and give a counterexample.

a. If $\lim_{x \to 0} f(x) = f(0)$, then $\lim_{x \to 0} \dfrac{f(x) - f(0)}{x}$ exists.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $|x|$

b. If the graph of $y = f(x)$ has an inflection point at $(0, f(0))$ and $f(0) \neq 0$, then the graph of $y = 1/f(x)$ has an inflection point at $(0, 1/f(0))$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $1+\sin x$

c. If f is continuous on $[-1, 1]$, then $\displaystyle\int_{-1}^{1} \left(f(x)\right)^2 dx = \left(\int_{-1}^{1} f(x)\, dx\right)^2$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ 1

d. If f is continuous on $(-\infty, \infty)$ and $\displaystyle\int_{-x}^{x} f(t)\, dt = 0$ for all $x > 0$, then f is an odd function.

☒ TRUE

☐ FALSE, because it does not hold for $f(x) =$

e. If f is integrable on $[-a, a]$ and $\displaystyle\int_{-a}^{a} f(t)\, dt = 0$ for all $a > 0$, then $f(0) = 0$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $\begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

4. For a positive continuous function f on $(-\infty, \infty)$, let

- $R(a)$ be the region between the graph of $y = f(x)$ and the x-axis for $x \leq a$,
- $A(a)$ be the area of $R(a)$, and
- $V(a)$ be the volume of the solid generated by revolving $R(a)$ about the x-axis.

a. Show that if $f(x) = e^{2x/\pi}$, then $V(a) = (A(a))^2$ for all a.

$$\left.\begin{aligned} A(a) &= \int_{-\infty}^{a} f(x)\,dx = \int_{-\infty}^{a} e^{2x/\pi}\,dx = \frac{\pi}{2} e^{2a/\pi} \\ V(a) &= \pi \int_{-\infty}^{a} f(x)^2\,dx = \pi \int_{-\infty}^{a} e^{4x/\pi}\,dx = \frac{\pi^2}{4} e^{4a/\pi} \end{aligned}\right\} \Rightarrow V(a) = A(a)^2 \text{ for all } a$$

because for a positive constant k:

$$\int_{-\infty}^{a} e^{kx}\,dx = \lim_{c \to -\infty} \int_{c}^{a} e^{kx}\,dx = \lim_{c \to -\infty} \left[\frac{e^{kx}}{k}\right]_{c}^{a} = \lim_{c \to -\infty} \frac{e^{ka} - \overset{\nearrow 0}{e^{kc}}}{k} = \frac{e^{ka}}{k}$$

b. Show that $f(x) = e^{2x/\pi}$ is the only positive continuous function on $(-\infty, \infty)$ for which $A(a)$ and $V(a)$ are finite and satisfy $V(a) = (A(a))^2$ for all a, and $f(0) = 1$.

$$V(a) = A(a)^2 \Rightarrow \pi \int_{-\infty}^{a} f(x)^2\,dx = \left(\int_{-\infty}^{a} f(x)\,dx\right)^2$$

$$\Rightarrow \pi \frac{d}{da} \int_{-\infty}^{a} f(x)^2\,dx = \frac{d}{da}\left(\int_{-\infty}^{a} f(x)\,dx\right)^2 \overset{\text{FTC1}}{\Longrightarrow} \pi f(a)^2 = 2\int_{-\infty}^{a} f(x)\,dx \cdot f(a)$$

$$\Rightarrow \pi f(a) = 2\int_{-\infty}^{a} f(x)\,dx \overset{\text{FTC1}}{\Longrightarrow} \pi f'(a) = 2 f(a)$$

$$\Rightarrow \frac{f'(a)}{f(a)} = \frac{2}{\pi} \Rightarrow \int \frac{f'(a)}{f(a)}\,da = \int \frac{2}{\pi}\,da \Rightarrow \ln|f(a)| = \frac{2}{\pi} a + C \text{ for all } a$$

$$\overset{\boxed{a=0}}{\Longrightarrow} 0 = \ln|\underbrace{f(0)}_{1}| = C$$

Hence $f(a) = e^{2a/\pi}$ for all a as $f(a) > 0$.

5. The towns A and B lie inland and the town C lies on the coast as shown in *Figures 1* and *2*. A port P will be built on the coast and connected to the towns A, B, C with straight roads. The cost of constructing the roads connecting P to A and P to B is 10^5 ฿/km, and the cost of constructing the road connecting P to C is $k \times 10^5$ ฿/km. The port will be built at the location for which the total cost of constructing these roads is the smallest.

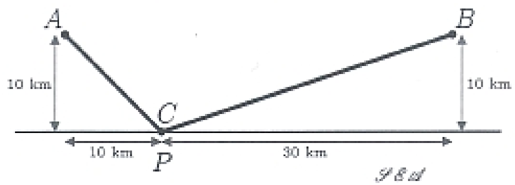

Figure 1

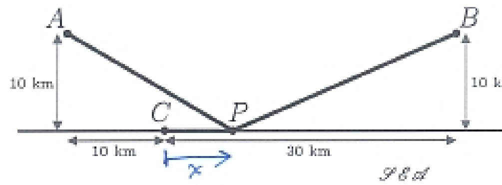

Figure 2

For some values of k the smallest total cost will be achieved when P is built at C, as in *Figure 1*; whereas for others it will be achieved, if at all, when P is built somewhere else, as in *Figure 2*. Determine the smallest value of k for which the lowest total cost is achieved by building P at C.

The total cost is given by

$$K(x) = (1^2 + (x+1)^2)^{1/2} + (1^2 + (3-x)^2)^{1/2} + k \cdot |x| \quad \text{for } -\infty < x < \infty$$

in units of 10^5 ฿, where the distances are measured in units of 10 km and x is the directed distance from C to the right. Then:

$$K'(x) = \begin{cases} (x+1)\cdot(1+(x+1)^2)^{-1/2} + (x-3)\cdot(1+(x-3)^2)^{-1/2} + k & \text{if } x > 0 \\ (x+1)\cdot(1+(x+1)^2)^{-1/2} + (x-3)\cdot(1+(x-3)^2)^{-1/2} - k & \text{if } x < 0 \end{cases}$$

1. If $k < \frac{3}{\sqrt{10}} - \frac{1}{\sqrt{2}}$, then $\lim_{x \to 0^+} K'(x) = -\left(\frac{3}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right) + k < 0$ and therefore K' is negative on some interval $(0,a)$. Then K is decreasing on $[0,a]$ as K is continuous. Therefore, the absolute minimum of K does not occur at $x=0$.

$$K''(x) = (1+(x+1)^2)^{-1/2} - (x+1)^2\cdot(1+(x+1)^2)^{-3/2} + (1+(x-3)^2)^{-1/2} - (x-3)^2\cdot(1+(x-3)^2)^{-3/2}$$
$$= (1+(x+1)^2)^{-3/2} + (1+(x-3)^2)^{-3/2} > 0 \quad \text{for all } x \neq 0$$

$\Rightarrow K'$ is increasing on $(-\infty, 0)$ and $(0, \infty)$. In particular:

2. If $k = \frac{3}{\sqrt{10}} - \frac{1}{\sqrt{2}}$, then $\lim_{x \to 0^+} K'(x) = 0 \Rightarrow K'(x) > 0$ for $x > 0$, and $\lim_{x \to 0^-} K'(x) = -2\cdot\left(\frac{3}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right) < 0 \Rightarrow K'(x) < 0$ for $x < 0$. Therefore, K has its absolute minimum at $x = 0$.

1 and 2 $\Longrightarrow \frac{3}{\sqrt{10}} - \frac{1}{\sqrt{2}}$ is the smallest value of k for which the absolute minimum of K occurs at C.