1. Evaluate the following limits by expressing the answers in terms of $A = \lim_{x\to 0} \dfrac{x - \sin x}{x^3}$.
[Do not use L'Hôpital's Rule!]

a. $\lim_{x\to 0} \dfrac{1 - x^2/2 - \cos x}{x^4} = \lim_{x\to 0} \dfrac{1-\cos x - \frac{x^2}{2}}{x^4} = \lim_{x\to 0} \dfrac{2\sin^2\frac{x}{2} - \frac{x^2}{2}}{x^4}$

$= 2\lim_{x\to 0} \dfrac{\sin^2\frac{x}{2} - \frac{x^2}{4}}{x^4} = 2\cdot\lim_{x\to 0} \dfrac{\sin\frac{x}{2} - \frac{x}{2}}{x^3}\cdot\lim_{x\to 0} \dfrac{\sin\frac{x}{2} + \frac{x}{2}}{x}$

$= -2\cdot\dfrac{1}{8}\cdot\underbrace{\lim_{x\to 0} \dfrac{\frac{x}{2} - \sin\frac{x}{2}}{(\frac{x}{2})^3}}_{=A}\cdot\left(\dfrac{1}{2}\underbrace{\lim_{x\to 0} \dfrac{\sin\frac{x}{2}}{\frac{x}{2}}}_{=1} + \dfrac{1}{2}\right) = -\dfrac{A}{4}$

b. $\lim_{x\to 0} \dfrac{x - \tan x}{x^3} = \lim_{x\to 0} \dfrac{x - \frac{\sin x}{\cos x}}{x^3} = \lim_{x\to 0} \dfrac{x\cos x - \sin x}{x^3}\cdot\lim_{x\to 0} \dfrac{1}{\cos x}$

$= \lim_{x\to 0} \dfrac{x\cos x - x + x - \sin x}{x^3}\cdot 1 = \lim_{x\to 0} \dfrac{\cos x - 1}{x^2} + \lim_{x\to 0} \dfrac{x - \sin x}{x^3}$

$= \lim_{x\to 0} \dfrac{-2\sin^2\frac{x}{2}}{x^2} + A = -\dfrac{1}{2}\Bigg(\underbrace{\lim_{x\to 0} \dfrac{\sin\frac{x}{2}}{\frac{x}{2}}}_{=1}\Bigg)^2 + A = A - \dfrac{1}{2}$

2a. Let P be the point on the graph of $y = x^5 - x^2$ with $x = 1$. Show that there is a point Q on the graph such that the tangent lines to the graph at the points P and Q are perpendicular to each other.

$y' = 5x^4 - 2x \Rightarrow$ (The slope of the tangent line at P) $= y'|_{x=1} = 3$

We want to show that there is a point Q on the graph such that the slope of the tangent line at Q is $-\frac{1}{3}$.

That is, we want to show that the equation $5x^4 - 2x \overset{*}{=} -\frac{1}{3}$ has a real solution.

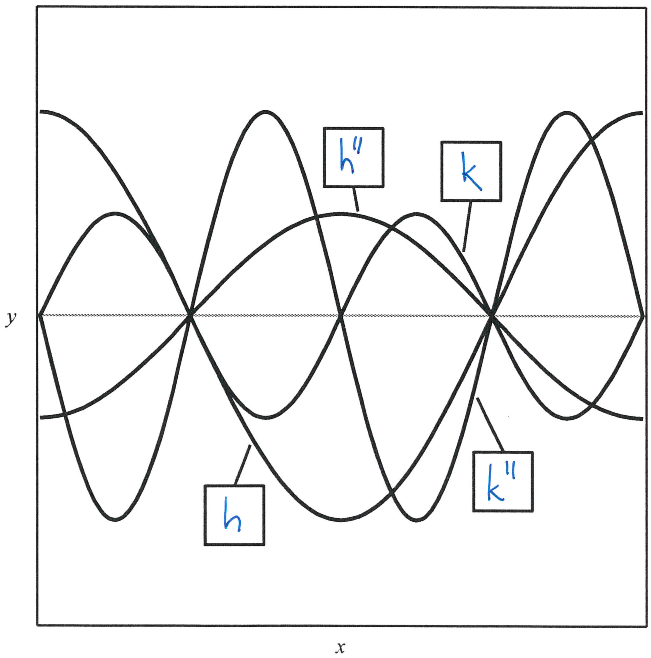
Let $f(x) = 15x^4 - 6x + 1$. Then $f(0) = 1 > 0$ and $f(\frac{1}{2}) = -\frac{17}{16} < 0$.

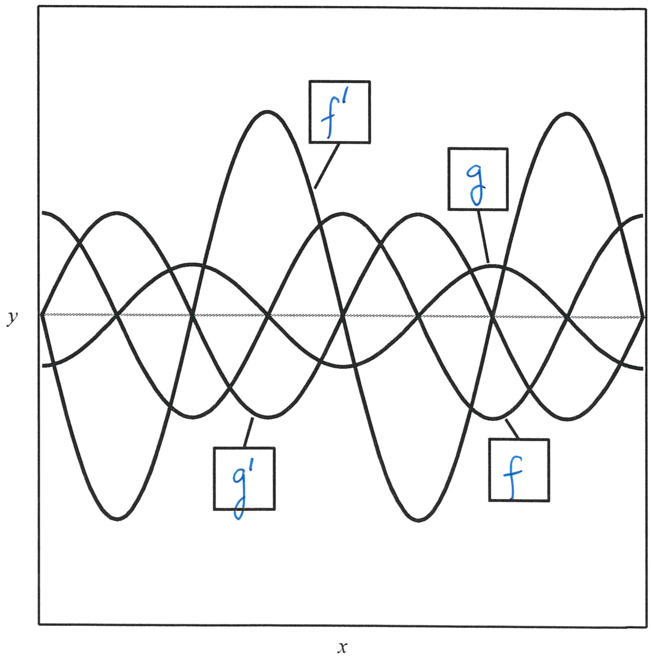
Since f is a polynomial, f is continuous on $[0, \frac{1}{2}]$.

Therefore, by IVT, there is a point c in $(0, \frac{1}{2})$ such that $f(c) = 0$.

Hence the equation $*$ has a real solution.

2b. In one of the following figures, the graphs of two functions f and g together with their derivatives f' and g' are shown; while in the other, the graphs of two functions h and k together with their second derivatives h'' and k'' are shown. Identify each by filling in the boxes with f, g, f', g', h, k, h'', and k''.





3. Find $\left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(\pi/2\,,\,\pi/6)}$ if y is a differentiable function of x satisfying the equation:

$$2\sin^2(x+y) = \sin x + \sin y$$

$$\Downarrow$$

$$1-\cos(2\cdot(x+y)) = \sin x + \sin y$$

$$\Downarrow \; d/dx$$

$$\sin(2\cdot(x+y))\cdot 2\cdot(1+y') = \cos x + \cos y\cdot y'$$

$$\Downarrow \; (x,y)=\left(\tfrac{\pi}{2},\tfrac{\pi}{6}\right)$$

$$\sin\left(\tfrac{4\pi}{3}\right)\cdot 2\cdot(1+y') = \cos\tfrac{\pi}{2} + \cos\tfrac{\pi}{6}\cdot y'$$

$$\Downarrow$$

$$-\frac{\sqrt{3}}{2}\cdot 2\cdot(1+y') = 0 + \frac{\sqrt{3}}{2}\cdot y'$$

$$\Downarrow$$

$$y' = -\frac{2}{3} \text{ at } (x,y)=\left(\tfrac{\pi}{2},\tfrac{\pi}{6}\right)$$

d/dx

$$\cos(2\cdot(x+y))\cdot 4\cdot(1+y')^2 + \sin(2\cdot(x+y))\cdot 2\cdot y'' = -\sin x - \sin y\cdot(y')^2 + \cos y\cdot y''$$

$$\Downarrow \; (x,y)=\left(\tfrac{\pi}{2},\tfrac{\pi}{6}\right),\; y'=-\tfrac{2}{3}$$

$$\cos\left(\tfrac{4\pi}{3}\right)\cdot 4\cdot\left(1+\left(-\tfrac{2}{3}\right)\right)^2 + \sin\left(\tfrac{4\pi}{3}\right)\cdot 2\cdot y'' = -\sin\tfrac{\pi}{2} - \sin\tfrac{\pi}{6}\cdot\left(-\tfrac{2}{3}\right)^2 + \cos\tfrac{\pi}{6}\cdot y''$$

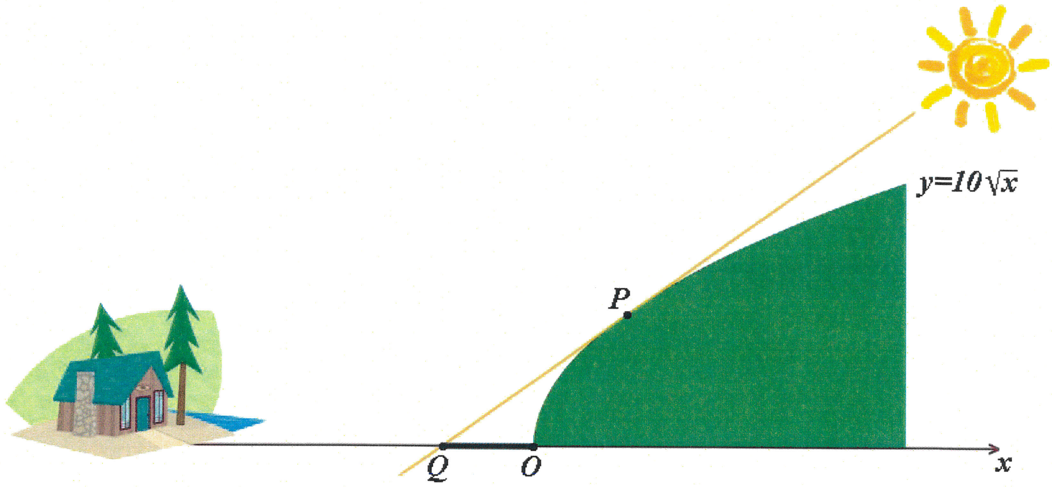
$$\Downarrow$$

$$-\frac{1}{2}\cdot 4\cdot\frac{1}{9} + \left(-\frac{\sqrt{3}}{2}\right)\cdot 2\cdot y'' = -1 - \frac{1}{2}\cdot\frac{4}{9} + \frac{\sqrt{3}}{2}\, y''$$

$$\Downarrow$$

$$y'' = \frac{2}{3\sqrt{3}} \text{ at } (x,y)=\left(\tfrac{\pi}{2},\tfrac{\pi}{6}\right)$$

4. You have a cabin on the negative x-axis. A hill whose height is given by $y = 10\sqrt{x}$ for $x \geq 0$ lies to the west along the positive x-axis. (All coordinates are measured in meters.) As the sun starts to set, the hill casts a shadow as shown in the figure. Determine how fast the shadow is approaching your cabin at the moment when the sunrays are making a 30° angle with the horizontal and this angle is decreasing at a rate of 1/4 °/min. Express your answer in units of meters per minute.



Let a be the x-coordinate of the point P where the sunray is tangent to the hill, and let θ be the angle between the sunray and the positive x-axis.

$$y' = \frac{10}{2\sqrt{x}} = \frac{5}{\sqrt{x}} \Rightarrow \tan\theta = (\text{The slope of the ray}) = y'\big|_{x=a} = \frac{5}{\sqrt{a}} \Rightarrow a = \frac{25}{\tan^2\theta}$$

The equation of the tangent line is: $y - 10\sqrt{a} = \frac{5}{\sqrt{a}} \cdot (x-a)$

Hence, $y=0 \Rightarrow -10\sqrt{a} = \frac{5}{\sqrt{a}} \cdot (x-a) \Rightarrow x = -a$ is the x-coordinate of Q.

$$|QO| = a = \frac{25}{\tan^2\theta} \Rightarrow \frac{d}{dt}|QO| = -2 \cdot \frac{25}{\tan^3\theta} \cdot \sec^2\theta \cdot \frac{d\theta}{dt}$$

When $\theta = 30° = \frac{\pi}{6}$ and $\frac{d\theta}{dt} = -\frac{1}{4}$ °/min $= -\frac{1}{4} \cdot \frac{\pi}{180}$ 1/min, this gives:

$$\frac{d}{dt}|QO| = -2 \cdot \frac{25}{\tan^3\frac{\pi}{6}} \cdot \sec^2\frac{\pi}{6} \cdot \left(-\frac{1}{4} \cdot \frac{\pi}{180}\right) = +2 \cdot \frac{25}{\left(\frac{1}{\sqrt{3}}\right)^3} \cdot \left(\frac{2}{\sqrt{3}}\right)^2 \cdot \frac{1}{4} \cdot \frac{\pi}{180} = \frac{5\pi}{6\sqrt{3}} \text{ m/min}$$

The shadow is approaching the cabin with a speed of $\frac{5\pi}{6\sqrt{3}}$ m/min at that moment.