Bilkent University

Quiz # 11
Math 101-Section **13** Calculus I
20 December 2018, Thursday
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Solution Key

Q-1) Evaluate $\displaystyle\int_0^{1/\sqrt{2}} x^2\sqrt{1-x^2}\,dx$. You may use the following identities.

$$\int \sin^2\theta\,d\theta = \frac{\theta}{2}-\frac{1}{2}\sin\theta\cos\theta+C,\quad \int \sin^4\theta\,d\theta = \frac{3\theta}{8}-\frac{1}{4}\sin^3\theta\cos\theta-\frac{3}{8}\sin\theta\cos\theta+C,\quad \int \sin^6\theta\,d\theta = \frac{5\theta}{16}-\frac{1}{6}\sin^5\theta\cos\theta-\frac{5}{24}\sin^3\theta\cos\theta-\frac{5}{16}\sin\theta\cos\theta+C$$

Solution:

We use trigonometric substitution $x=\sin\theta$. Then $dx=\cos\theta\,d\theta$, $x^2\sqrt{1-x^2}=\sin^2\theta\cos\theta$ and the integral becomes from $\theta=0$ to $\theta=\pi/4$.

$$\begin{aligned}
\int_0^{1/\sqrt{2}} x^2\sqrt{1-x^2}\,dx &= \int_0^{\pi/4}\sin^2\theta\cos^2\theta\,d\theta = \int_0^{\pi/4}\sin^2\theta(1-\sin^2\theta)\,d\theta \\
&= \int_0^{\pi/4}\sin^2\theta\,d\theta - \int_0^{\pi/4}\sin^4\theta\,d\theta \\
&= \left(\frac{\theta}{2}-\frac{1}{2}\sin\theta\cos\theta\Big|_0^{\pi/4}\right) - \left(\frac{3\theta}{8}-\frac{1}{4}\sin^3\theta\cos\theta-\frac{3}{8}\sin\theta\cos\theta\Big|_0^{\pi/4}\right) \\
&= \frac{\pi}{32}.
\end{aligned}$$