1. In each of the following, if the given statement is true for all functions f that are defined on $(-\infty, \infty)$, then mark the ☐ to the left of TRUE with a ✗; otherwise, mark the ☐ to the left of FALSE with a ✗ and give a counterexample.

a. If $f(2x) = f(x)$ for all x, then f is constant on $(-\infty, \infty)$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$

b. If f is continuous on $(-\infty, \infty)$, then f has a derivative on $(-\infty, \infty)$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) = |x|$

c. If f is continuous on $(-\infty, \infty)$, then f has an antiderivative on $(-\infty, \infty)$.

☒ TRUE

☐ FALSE, because it does not hold for $f(x) =$

d. If f is continuous on $(-\infty, \infty)$, then $\int f(x)\, dx = \frac{1}{2} f(x)^2 + C$ on $(-\infty, \infty)$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) = 1$

e. If $\int_{-1}^{1} f(x)\, dx = 0$, then $\int_{-1}^{1} f(x)^3\, dx = 0$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) = \begin{cases} -1 & \text{if } x < \frac{1}{2} \\ 3 & \text{if } x \geq \frac{1}{2} \end{cases}$

2. Evaluate the following limits:

a. $\lim_{x\to 0} \dfrac{1 - x^2/2 - \cos x}{x^4} \overset{\text{L'H}}{=} \lim_{x\to 0} \dfrac{-x+\sin x}{4x^3} \overset{\text{L'H}}{=} \lim_{x\to 0} \dfrac{-1+\cos x}{12x^2} \overset{\text{L'H}}{=} \lim_{x\to 0} \dfrac{-\sin x}{24x} = -\dfrac{1}{24}$

$\dfrac{\sin x}{x} \to 1$

b. $\lim_{x\to 0} \dfrac{x - \tan x}{x^3} = \lim_{x\to 0} \dfrac{x\cos x - \sin x}{x^3 \cos x} \overset{\text{L'H}}{=} \lim_{x\to 0} \dfrac{\cos x - x\sin x - \cos x}{3x^2\cos x - x^3 \sin x}$

$= \lim_{x\to 0} \dfrac{-\dfrac{\sin x}{x}}{\underbrace{3\cos x}_{\to 1} - \underbrace{x\sin x}_{\to 0}} = -\dfrac{1}{3}$

$\dfrac{\sin x}{x} \to 1$

3. Evaluate the following integrals:

a. $\displaystyle\int_0^{\pi} \sin x \cos x \sin(\pi \cos x)\, dx = -\frac{1}{\pi^2}\int_{\pi}^{-\pi} t \sin t \, dt = \frac{1}{\pi^2}\int_{-\pi}^{\pi} t \sin t \, dt$

$t = \pi \cos x$
$dt = -\pi \sin x \, dx$

$$= \frac{1}{\pi^2}\left(\Big[-t\cos t\Big]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos t \, dt\right) = \frac{1}{\pi^2}\left(-\pi\cos\pi - \pi\cos(-\pi) + \Big[\sin t\Big]_{-\pi}^{\pi}\right)$$

$u = t \Rightarrow du = dt$
$dv = \sin t \, dt \Rightarrow v = -\cos t$

$$= \frac{1}{\pi^2}\left(\pi + \pi + \sin(\pi) - \sin(-\pi)\right) = \frac{2}{\pi}$$

b. $\displaystyle\int \frac{\tan\theta}{(1-\tan^2\theta)^2}\, d\theta = \int \frac{\sin\theta \cos^3\theta}{(\cos^2\theta - \sin^2\theta)^2}\, d\theta = \int \frac{\frac{1}{2}\sin 2\theta \cdot \cos^2\theta}{\cos^2 2\theta}\, d\theta$

$$= \frac{1}{4}\int \frac{\sin 2\theta \cdot (1+\cos 2\theta)}{\cos^2 2\theta}\, d\theta = -\frac{1}{8}\int \left(\frac{1}{u^2} + \frac{1}{u}\right) du$$

$u = \cos 2\theta$
$du = -2\sin 2\theta \, d\theta$

$$= \frac{1}{8}\left(\frac{1}{u} - \ln|u|\right) + C' = \frac{1}{8\cos 2\theta} - \frac{1}{8}\ln|\cos 2\theta| + C$$

4. The coordinates (x_c, y_c) of the *center* of a region R contained in the first quadrant of the xy-plane are defined by

$$x_c = \frac{W}{2\pi A} \quad \text{and} \quad y_c = \frac{V}{2\pi A}$$

where

- A is the area of R,
- V is the volume of the solid generated by revolving R about the x-axis, and
- W is the volume of the solid generated by revolving R about the y-axis.

Let R be the region lying between the graph of $y = (x^2+1)^{-3/2}$ and the x-axis for $x \geq 0$. Compute one of the coordinates ☑ x_c or ☐ y_c of the center of R.

[Indicate the one you are computing by putting a ✓ in the ☐ to the left of it.]

$$A = \int_0^\infty (x^2+1)^{-3/2}\,dx = \int_0^{\pi/2} (\tan^2\theta+1)^{-3/2}\sec^2\theta\,d\theta = \int_0^{\pi/2} (\sec^2\theta)^{-3/2}\sec^2\theta\,d\theta$$

$$\boxed{\begin{array}{l} x = \tan\theta \\ dx = \sec^2\theta\,d\theta \end{array}}$$

$$= \int_0^{\pi/2} \cos\theta\,d\theta = \sin\theta\Big]_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1$$

$$W = 2\pi\int_0^\infty (\text{radius})\cdot(\text{height})\,dx = 2\pi\int_0^\infty x\cdot(x^2+1)^{-3/2}\,dx$$

$$= 2\pi\int_0^{\pi/2} \tan\theta\cdot(\sec^2\theta)^{-3/2}\sec^2\theta\,d\theta = 2\pi\int_0^{\pi/2}\sin\theta\,d\theta = 2\pi\cdot\Big[-\cos\theta\Big]_0^{\pi/2}$$

$$\boxed{\begin{array}{l} x = \tan\theta \\ dx = \sec^2\theta\,d\theta \end{array}}$$

$$= 2\pi\left(-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right) = 2\pi$$

$$x_c = \frac{W}{2\pi A} = \frac{2\pi}{2\pi\cdot 1} = 1$$

5. Find all values of the constant k for which the function $f(x) = \dfrac{k}{(e^x + e^{-x})^2}$ satisfies the equation

$$f'(x) = f(x) \int_0^x f(t)\, dt$$

for all x.

$$\int \frac{1}{(e^t+e^{-t})^2}\,dt \overset{\boxed{\substack{u=e^t\\ du=e^t\,dt}}}{=} \int \frac{1}{(u+u^{-1})^2}\cdot\frac{du}{u} = \int \frac{u}{(u^2+1)^2}\,du = -\frac{1}{2}\cdot\frac{1}{u^2+1} + C = -\frac{1}{2}\cdot\frac{1}{e^{2t}+1} + C$$

$$\Rightarrow \int_0^x f(t)\,dt = -\frac{k}{2}\left[\frac{1}{e^{2t}+1}\right]_0^x = -\frac{k}{2}\cdot\left(\frac{1}{e^{2x}+1} - \frac{1}{2}\right) = \frac{k}{4}\cdot\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

$$f'(x) = -2k\cdot\frac{1}{(e^x+e^{-x})^3}\cdot(e^x-e^{-x})$$

Hence:

$$f'(x) = f(x)\int_0^x f(t)\,dt \quad \text{for all } x$$

$$\Updownarrow$$

$$-2k\cdot\frac{1}{(e^x+e^{-x})^3}\cdot(e^x-e^{-x}) = \frac{k}{(e^x+e^{-x})^2}\cdot\frac{k}{4}\cdot\frac{e^x-e^{-x}}{(e^x+e^{-x})} \quad \text{for all } x$$

$$\Updownarrow$$

$$-2k = \frac{k^2}{4}$$

$$\Updownarrow$$

$$k=0 \quad \text{or} \quad k=-8$$