1. A continuous function f on $(-\infty, \infty)$ satisfies the following conditions:

❶ $f(0) = 6\,,\;\; f(3) = 0$

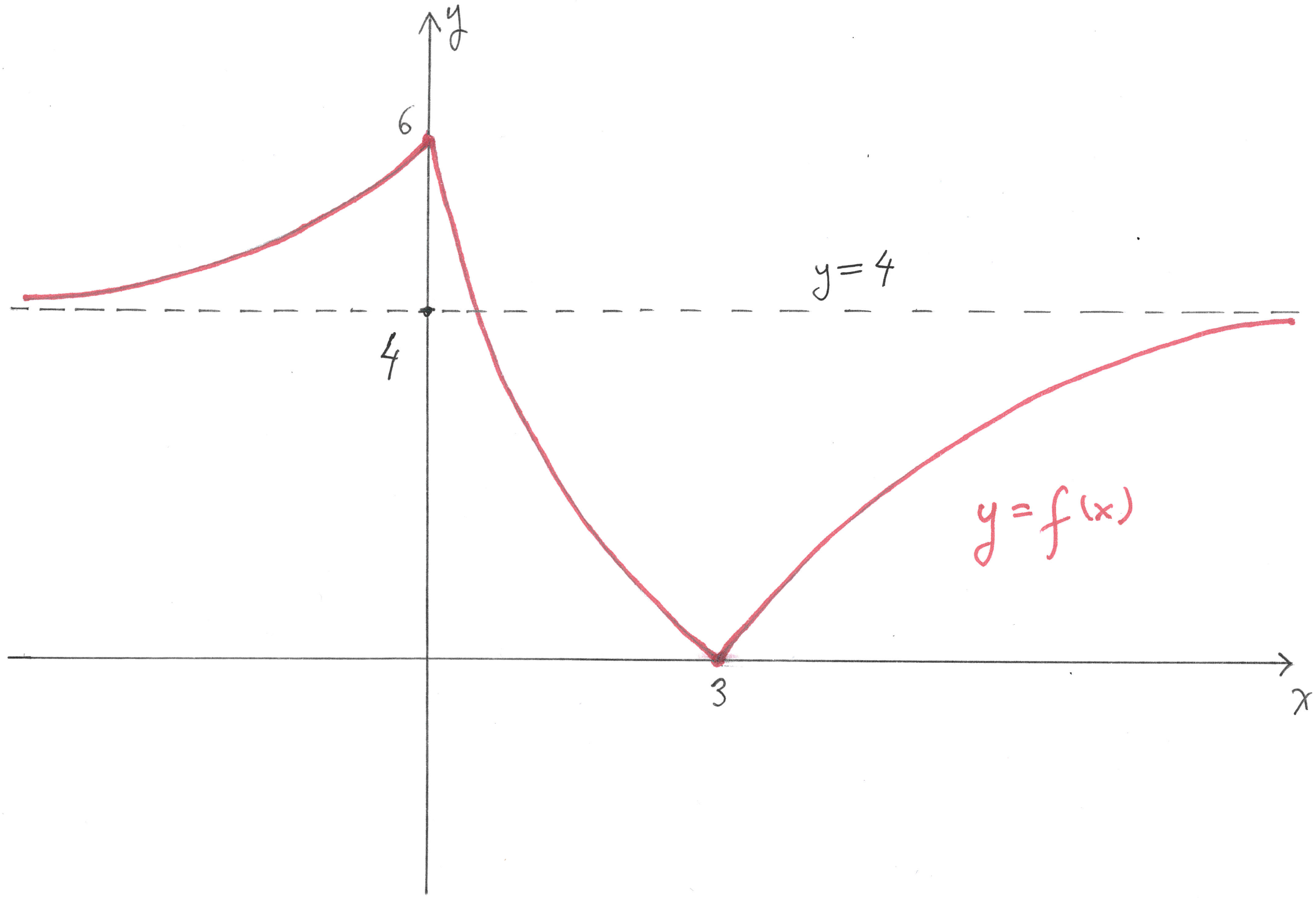
❷ $f'(x) > 0$ for $x < 0$ and for $3 < x\,;\;\; f'(x) < 0$ for $0 < x < 3$

❸ $f''(x) > 0$ for $x < 0$ and for $0 < x < 3\,;\;\; f''(x) < 0$ for $3 < x$

❹ $\lim\limits_{x \to -\infty} f(x) = 4\,,\;\; \lim\limits_{x \to \infty} f(x) = 4$

❺ $\lim\limits_{x \to 0^-} f'(x) = 1\,,\;\; \lim\limits_{x \to 0^+} f'(x) = -5\,;\;\; \lim\limits_{x \to 3^-} f'(x) = -4/5\,,\;\; \lim\limits_{x \to 3^+} f'(x) = 4/5$

a. Sketch the graph of $y = f(x)$ making sure that all important features are clearly shown.



b. Fill in the boxes to make the following a true statement. No explanation is required.

The function $f(x) = \dfrac{|x-a|}{b|x|+c}$ satisfies the conditions ❶-❺ if a, b and c are chosen as

$a = \boxed{3}$, $b = \boxed{\frac{1}{4}}$ and $c = \boxed{\frac{1}{2}}$.

Since $f(3) = 0$, we immediately have $a = 3$. Hence we have

$$f(x) = \begin{cases} \dfrac{x-3}{bx+c} & x \geq 3, \\[2ex] \dfrac{3-x}{bx+c} & 0 \leq x \leq 3, \\[2ex] \dfrac{3-x}{-bx+c} & x \leq 0. \end{cases}$$

On the other hand $f(0) = 6$ gives $\dfrac{3}{c} = 6$, or equivalently $c = \dfrac{1}{2}$.

Finally $\lim\limits_{x \to \infty} f(x) = 4$ gives $\dfrac{1}{b} = 4$, or equivalently $b = \dfrac{1}{4}$.

2. Suppose f is a twice-differentiable function on $(-\infty, \infty)$ satisfying the following conditions:

❶ $x = 4$ and $x = 11$ are the only critical points of f in the interval $(0, 15)$.

❷ $f(0) = 1,\ f(4) = -2,\ f(11) = 4,\ f(15) = -1$.

❸ $f'(0) = -1,\ f'(15) = -2$.

❹ $|f''(x)| \leq 1$ for all x in the interval $[0, 15]$.

a. Let $g(x) = 3f(x) - (f'(x))^2$. Show that the information given above is sufficient to determine the absolute maximum and minimum values of the function g on the interval $[0, 15]$, and find them.

$$g'(x) = 3f'(x) - 2f'(x)f''(x) = f'(x)\cdot(3 - 2f''(x))$$

$$g'(x) = 0 \Rightarrow f'(x) = 0 \quad \text{or} \quad f''(x) = \frac{3}{2}$$

$f'(x) = 0 \Downarrow$ $x = 4,\ x = 11$ for $0 < x < 15$

This equation has no solution in $(0, 15)$ as $f''(x) \leq 1$ by ④.

Critical points:

$x = 4 \Rightarrow g(4) = 3f(4) - f'(4)^2 = 3\cdot(-2) - 0^2 = -6$

$x = 11 \Rightarrow g(11) = 3f(11) - f'(11)^2 = 3\cdot 4 - 0^2 = 12$

Endpoints:

$x = 0 \Rightarrow g(0) = 3f(0) - f'(0)^2 = 3\cdot 1 - (-1)^2 = 2$

$x = 15 \Rightarrow g(15) = 3f(15) - f'(15)^2 = 3\cdot(-1) - (-2)^2 = -7$

Abs max and min of g on $[0, 15]$ are 12 and -7.

b. Show that there is a point c in the interval $(0, 15)$ such that $f''(c) = -1/2$.

f is twice-differentiable $\Rightarrow f'$ is differentiable and continuous everywhere.

$x = 11$ is a critical point of f and f' is defined at $x = 11 \Rightarrow f'(11) = 0$.

Applying MVT to f' on $[11, 15]$, we conclude that there is a point c in $(11, 15)$ such that:

$$f''(c) = \frac{f'(15) - f'(11)}{15 - 11} = \frac{-2 - 0}{4} = -\frac{1}{2}$$

3a. The slope of the tangent line at each point (x, y) on the graph of a differentiable function $y = f(x)$ is proportional to $x^2 - 5$. If $f(1) = 1$ and $f(3) = 3$, find $f(2)$.

$f'(x) = k\cdot(x^2-5)$ for some constant k

$\Downarrow$

$f(x) = \int f'(x)\,dx = k\int (x^2-5)\,dx = k\cdot\left(\frac{x^3}{3} - 5x\right) + C$

$$\left.\begin{cases} 1 = f(1) = k\cdot\left(\frac{1}{3}-5\right) + C = -\frac{14}{3}k + C \\ 3 = f(3) = k\cdot(9-15) + C = -6k + C \end{cases}\right\} \Rightarrow -2 = \frac{4}{3}k \Rightarrow k = -\frac{3}{2}$$

$\Downarrow$

$C = -6$

$f(x) = -\frac{3}{2}\cdot\left(\frac{1}{3}x^3 - 5x\right) - 6 \Rightarrow f(2) = -\frac{3}{2}\cdot\left(\frac{8}{3} - 10\right) - 6 = 5$

3b. Suppose that a continuous function g satisfies:

$$\int_0^3 g(x)\,dx = 7 \qquad \text{and} \qquad \int_0^6 g(2x)\,dx = 5$$

Find $\int_1^2 x\,g(3x^2)\,dx$.

$$5 = \int_0^6 g(2x)\,dx \underset{\boxed{\substack{u=2x \\ du = 2dx}}}{=} \int_0^{12} g(u)\cdot\frac{1}{2}\,du \Rightarrow \int_0^{12} g(u)\,du = 10$$

$$\int_1^2 x\,g(3x^2)\,dx \underset{\boxed{\substack{u=3x^2 \\ du = 6x\,dx}}}{=} \int_3^{12} g(u)\cdot\frac{1}{6}\,du = \frac{1}{6}\left(\int_0^{12} g(u)\,du - \int_0^3 g(u)\,du\right)$$

$$= \frac{1}{6}\cdot(10-7) = \frac{1}{2}$$

4. Evaluate the following integrals.

a. $\displaystyle\int_0^{\pi/4} \frac{\sec^2 x}{(2\tan x+1)^2}\,dx = \int_1^3 \frac{1}{u^2}\cdot\frac{1}{2}\,du = \left[-\frac{1}{2u}\right]_1^3 = -\frac{1}{6}+\frac{1}{2} = \frac{1}{3}$

$u = 2\tan x + 1$

$du = 2\sec^2 x\,dx$

b. $\displaystyle\int_0^{\pi/4} \frac{\sin x}{(2\sin x+\cos x)^3}\,dx = \int_0^{\pi/4} \frac{\tan x\cdot\sec^2 x}{(2\tan x+1)^3}\,dx = \int_1^3 \frac{(u-1)/2}{u^3}\cdot\frac{1}{2}\,du$

$u = 2\tan x + 1$

$du = 2\sec^2 x\,dx$

$\displaystyle = \frac{1}{4}\int_1^3 (u^{-2}-u^{-3})\,du = \frac{1}{4}\left[\frac{u^{-1}}{-1} - \frac{u^{-2}}{-2}\right]_1^3 = \frac{1}{4}\left(-\frac{1}{3}+\frac{1}{18}+1-\frac{1}{2}\right) = \frac{1}{18}$