1. A differentiable function f and the polynomials p, q, r, s satisfy

$$(*) \qquad -x^2+\frac{9}{2}x-\frac{1}{2}=p(x)\le f(x)\le q(x)=x^2+2 \quad \text{for all } x<1\,,$$

and

$$(**) \qquad -\frac{3}{2}x^3+6x^2-6x+\frac{9}{2}=r(x)\le f(x)\le s(x)=-x^3+5x^2-5x+4 \quad \text{for all } x>1\,.$$

a. Find $f(1)$.

$$\left.\begin{aligned} \lim_{x\to 1^-} p(x) &= -1^2+\tfrac{9}{2}\cdot 1-\tfrac{1}{2}=3 \\ \lim_{x\to 1^-} q(x) &= 1^2+2=3 \end{aligned}\right\} \Rightarrow \lim_{x\to 1^-} f(x)=3 \text{ by } (*) \text{ and the Squeeze Theorem.}$$

f is differentiable $\Rightarrow$ f is continuous $\Rightarrow f(1)=\lim_{x\to 1^-} f(x)=3$.

b. Find $f'(1)$.

For $h<0$:

$$(*) \Rightarrow f(1+h)\le q(1+h) \Rightarrow f(1+h)-f(1)\le q(1+h)-q(1) \Rightarrow \frac{f(1+h)-f(1)}{h}\ge \frac{q(1+h)-q(1)}{h}$$

$$\Rightarrow f'(1)=\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}\ge \lim_{h\to 0^-}\frac{q(1+h)-q(1)}{h}=q'(1)=2x\Big|_{x=1}=2$$

For $h>0$:

$$(**) \Rightarrow f(1+h)\le s(1+h) \Rightarrow f(1+h)-f(1)\le s(1+h)-s(1) \Rightarrow \frac{f(1+h)-f(1)}{h}\le \frac{s(1+h)-s(1)}{h}$$

$$\Rightarrow f'(1)=\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}\le \lim_{h\to 0^+}\frac{s(1+h)-s(1)}{h}=s'(1)=(-3x^2+10x-5)\Big|_{x=1}=2$$

So: $f'(1)\ge 2$ and $f'(1)\le 2 \Rightarrow f'(1)=2$

c. Show that there is a point c such that $f'(c)=0$.

$$(**) \Rightarrow \begin{cases} f(2)\ge r(2)=-\frac{3}{2}\cdot 2^3+6\cdot 2^2-6\cdot 2+\frac{9}{2}=\frac{9}{2}>3 \\ f(4)\le s(4)=-4^3+5\cdot 4^2-5\cdot 4+4=0<3 \end{cases}$$

f is differentiable $\Rightarrow$ f is continuous on $[2,4]$

$\Rightarrow$ By IVT, there is c_0 in $(2,4)$ such that $f(c_0)=3$.

Since f is continuous on $[1,c_0]$ and differentiable on $(1,c_0)$, by Rolle's Theorem, there is c in $(1,c_0)$ such that $f'(c)=0$.

2. Find $\left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(1/2\,,1)}$ if y is a differentiable function of x satisfying the equation:

$$\sin(\pi x y) = \frac{1}{x} - \frac{1}{y}$$

$$\Downarrow d/dx$$

$$\cos(\pi x y)\cdot\pi\cdot(y + x y') = -\frac{1}{x^2} + \frac{1}{y^2}\,y'$$

$$\boxed{(x,y) = (\tfrac{1}{2},1)}$$

$$\underbrace{\cos(\tfrac{\pi}{2})}_{0}\cdot\pi\cdot(1 + \tfrac{1}{2}\cdot y') = -4 + y'$$

$$\Downarrow$$

$$\boxed{y' = 4 \text{ at } (x,y) = (\tfrac{1}{2},1)}$$

d/dx

$$-\sin(\pi x y)\cdot(\pi\cdot(y + x y'))^2 + \cos(\pi x y)\cdot\pi\cdot(y' + y' + x y'')$$

$$= \frac{2}{x^3} - \frac{2}{y^3}\cdot(y')^2 + \frac{1}{y^2}\cdot y''$$

$$\boxed{(x,y) = (\tfrac{1}{2},1),\ y' = 4}$$

$$-\underbrace{\sin(\tfrac{\pi}{2})}_{1}\cdot(\pi\cdot(1 + \tfrac{1}{2}\cdot 4))^2 + \underbrace{\cos(\tfrac{\pi}{2})}_{0}\cdot\pi\cdot(4 + 4 + \tfrac{1}{2}y'')$$

$$= 16 - 2\cdot 16 + y''$$

$$\Downarrow$$

$$\boxed{y'' = 16 - 9\pi^2 \text{ at } (x,y) = (\tfrac{1}{2},1)}$$

3. As you sit on a shore, you notice a duck swimming towards its nest. At that moment, the duck is 7 m away from its nest and swimming with a speed of 2 m/s, and the distance between you and the duck is 8 m and increasing at a rate of 1 m/s. Determine how far you are from the duck's nest.

Let the lengths a, b, c, and the angle A be as shown in the figure.

By the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Downarrow d/dt$$

$$a\frac{da}{dt} = b\frac{db}{dt} - \frac{db}{dt}c\cos A$$

$$\boxed{a = 8\,m,\ \frac{da}{dt} = 1\,m/s,\ b = 7\,m,\ \frac{db}{dt} = -2\,m/s}$$

$$8^2 = 7^2 + c^2 - 2\cdot 7\cdot c\cos A$$

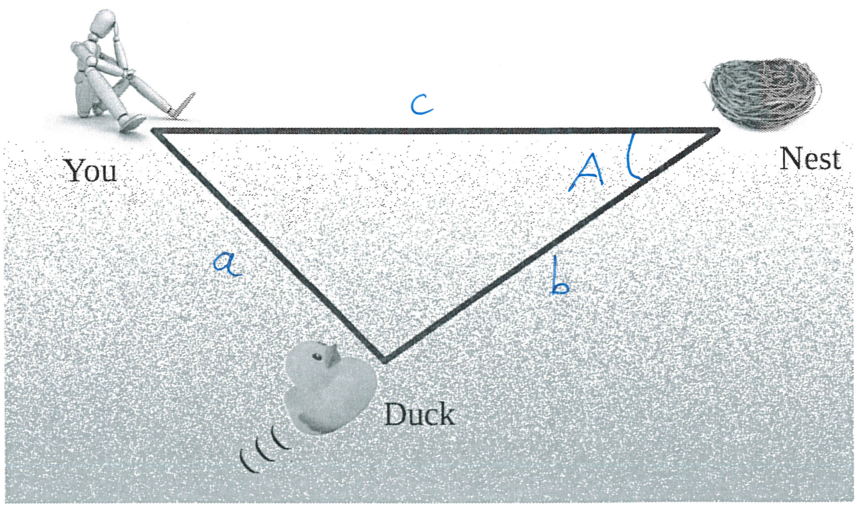
$$8\cdot 1 = 7\cdot(-2) - (-2)\cdot c\cos A \Rightarrow c\cos A = 11$$

$$c^2 = 169$$

$$\Downarrow$$

$$c = 13\,m$$

$\Rightarrow$ You are 13 m away from the duck's nest.



4. Find the absolute maximum and minimum values of $f(x) = -x^4 + \frac{21}{2}x^2 - 10x$ on the interval $[-3, 3]$.

$$f'(x) = -4x^3 + 21x - 10 = -(x-2)\underbrace{(4x^2+8x-5)}_{(2x-1)(2x+5)} = 0 \Rightarrow x = 2, x = \frac{1}{2}, x = -\frac{5}{2}$$

$$\boxed{f'(2) = 0}$$

Critical points:

$x = 2 \Rightarrow f(2) = -2^4 + \frac{21}{2} \cdot 2^2 - 10 \cdot 2 = 6$

$x = \frac{1}{2} \Rightarrow f(\frac{1}{2}) = -(\frac{1}{2})^4 + \frac{21}{2} \cdot (\frac{1}{2})^2 - 10 \cdot \frac{1}{2} = -\frac{39}{16}$

$x = -\frac{5}{2} \Rightarrow f(-\frac{5}{2}) = -(-\frac{5}{2})^4 + \frac{21}{2} \cdot (-\frac{5}{2})^2 - 10 \cdot (-\frac{5}{2}) = \frac{825}{16}$ ← largest

Endpoints:

$x = -3 \Rightarrow f(-3) = -(-3)^4 + \frac{21}{2} \cdot (-3)^2 - 10 \cdot (-3) = \frac{87}{2}$

$x = 3 \Rightarrow f(3) = -3^4 + \frac{21}{2} \cdot 3^2 - 10 \cdot 3 = -\frac{33}{2}$ ← smallest

Absolute maximum is $\frac{825}{16}$.

Absolute minimum is $-\frac{33}{2}$.

5. A function f, which is defined and continuous for all $x \neq 2$, satisfies the following conditions:

① $f(0) = 2$, $f(\sqrt{6}-2) = (22+8\sqrt{6})/25$

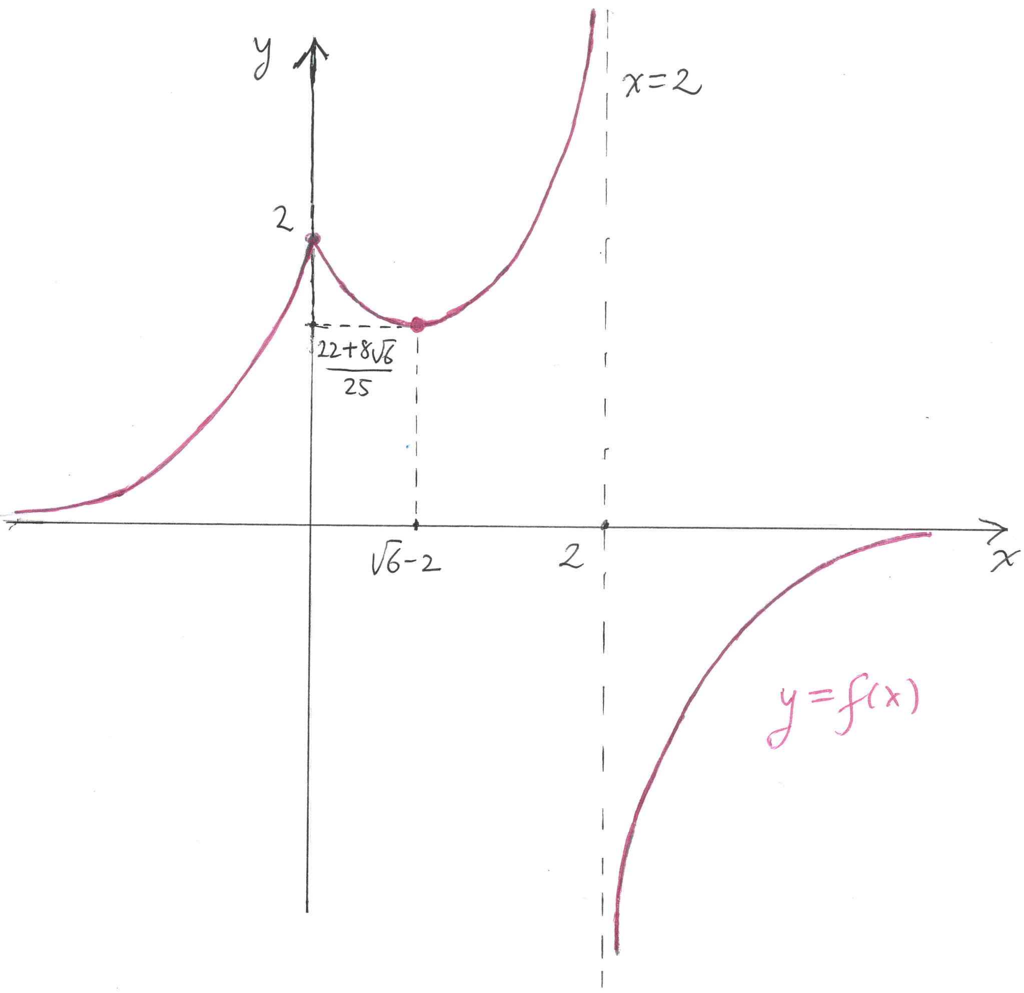
② $\lim_{x\to 2^-} f(x) = \infty$, $\lim_{x\to 2+} f(x) = -\infty$, $\lim_{x\to -\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = 0$

③ $f'(x) > 0$ for $x < 0$, and for $x > \sqrt{6}-2$ and $x \neq 2$; and $f'(x) < 0$ for $0 < x < \sqrt{6}-2$

④ $\lim_{x\to 0^-} f'(x) = 4$, $\lim_{x\to 0+} f'(x) = -2$

⑤ $f''(x) > 0$ for $x < 2$ and $x \neq 0$, $f''(x) < 0$ for $x > 2$

a. Sketch the graph of $y = f(x)$ making sure that all important features are clearly shown.



b. Fill in the boxes to make the following a true statement. No explanation is required.

The function $f(x) = \dfrac{ax+b}{x^2+c|x|+d}$ satisfies the conditions ①-⑤ at all points in its domain if a, b, c and d are chosen as

$a = \boxed{-1}$, $b = \boxed{-2}$, $c = \boxed{-\frac{3}{2}}$ and $d = \boxed{-1}$.