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Math 102 Calculus – Final Exam–Solutions

Q-1) Calculate the limit $\lim_{x\to 0} \dfrac{120(\sin x - x) + 20x^3 - x^5}{3x^7 + x^9}$.

Solution: Note that $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} - \cdots$. Then

$120(\sin x - x) = -20x^3 + x^5 - \dfrac{1}{42}x^7 + \dfrac{1}{3024}x^9 - \cdots$, and

$$\lim_{x\to 0} \frac{120(\sin x - x) + 20x^3 - x^5}{3x^7 + x^9} = \lim_{x\to 0} \frac{-1/42 + x^2/3024 - \cdots}{3 + x^2} = -\frac{1}{126}.$$

Q-2) Evaluate the integral $\displaystyle\int_0^2 \int_{x^2}^4 \frac{x \sin\left(\frac{\pi}{4}y\right)}{y} dy\, dx$.

Solution: The integral cannot be evaluated in this order. Changing the order of integration we get

$$\begin{aligned}
\int_0^2 \int_{x^2}^4 \frac{x \sin\left(\frac{\pi}{4}y\right)}{y} dy\, dx &= \int_0^4 \int_0^{\sqrt{y}} \frac{\sin(y\pi/4)}{y}\, x\, dx\, dy \\
&= \int_0^4 \left(\frac{\sin(y\pi/4)}{y}\right) \left(\frac{1}{2}x^2|_0^{\sqrt{y}}\right) dy \\
&= \frac{1}{2}\int_0^4 \sin(y\pi/4) dy \\
&= \frac{2}{\pi}(-\cos(y\pi/4)|_0^4) \\
&= \frac{4}{\pi}.
\end{aligned}$$

Q-3) Find the value of the sum $\displaystyle\sum_{n=0}^{\infty} \frac{8}{(n+3)(n+5)(n+7)}$.

Solution: By partial fractions technique we get

$$\frac{8}{(n+3)(n+5)(n+7)} = \frac{1}{n+3} - \frac{2}{n+5} + \frac{1}{n+7}.$$

By using telescoping sums we find

$$\begin{aligned}
S_n &= a_0 + a_1 + \cdots + a_n \\
&= \frac{13}{60} + \frac{1}{n+6} + \frac{1}{n+7} - \frac{1}{n+4} - \frac{1}{n+5}
\end{aligned}$$

from where it follows that the sum is $\lim_{n\to\infty} S_n = \dfrac{13}{60}$.

Q-4) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n\, n}$.

Solution: Letting $a_n = \frac{(x-1)^n}{2^n\, n}$ and using ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1}\frac{|x-1|}{2} \to \frac{|x-1|}{2} \quad \text{as} \quad n \to \infty,$$

or equivalently $-1 < x < 3$. Next we examine the end points:
When $x = -3$, $a_n = 1/n$ and the series diverges.
When $x = -1$, $a_n = (-1)^n/n$ and the series converges.
The interval of convergence is $[-1, 3)$.
To recognize the function observe that $f'(x) = 1/(3-x)$. Integrating f' and deciding on the integration factor using the fact that $f(1) = 0$, we find that $f(x) = \ln 2 - \ln(3-x)$.

Q-5) Find the volume of the region which remains inside the cyclinder $x^2 + y^2 = 2y$, and is bounded from above by the paraboloid surface $x^2 + y^2 + z = 1$ and from below by the plane $z = 0$.

Solution: Let V denote the required volume. Using the symmetry of the set up and changing to cylindrical coordinates we get

$$\frac{1}{2}V = \int_0^{\pi/6}\int_0^{2\sin\theta}(1-r^2)r dr\, d\theta + \int_{\pi/6}^{\pi/2}\int_0^1 (1-r^2)r dr\, d\theta.$$

Next we compute these integrals separately:

$$\begin{aligned}
\int_0^{\pi/6}\int_0^{2\sin\theta}(1-r^2)r dr\, d\theta &= \int_0^{\pi/6}\left(\frac{1}{2}r^2 - \frac{1}{4}r^4\Big|_0^{2\sin\theta}\right) d\theta \\
&= 2\int_0^{\pi/6}\sin^2\theta d\theta - 4\int_0^{\pi/6}\sin^4\theta d\theta \\
&= \frac{3\sqrt{3}}{16} - \frac{\pi}{12}
\end{aligned}$$

where for the last integral we used the formula given on the last page of the exam booklet.
Next we calculate the second integral:

$$\begin{aligned}
\int_{\pi/6}^{\pi/2}\int_0^1 (1-r^2)r dr\, d\theta &= \left(\frac{\pi}{2} - \frac{\pi}{6}\right)\left(\frac{1}{2} - \frac{1}{4}\right) \\
&= \frac{\pi}{12}.
\end{aligned}$$

It then follows that

$$V = \frac{3\sqrt{3}}{8}.$$
