Date: 21 July 2004, Wednesday NAME:..
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Time: 10:40-11:40 STUDENT NO:..

Math 102 Calculus II – QUIZ II – Solutions

Q-1) Let $\vec{r}(t) = (t, \ln\cos t)$, $t \in [0, \pi/4]$ be the curve C.
Find **T, N**, κ and the length of the curve C.

Solution:
$v = (1, -\tan t)$, $|v| = \sqrt{1+\tan^2 t} = \sec t$.

Length$=\displaystyle\int_0^{\pi/4} |v|\ dt = \int_0^{\pi/4} \sec t\ dt = \left(\ln|\sec t + \tan t||_0^{\pi/4}\right) = \ln(1+\sqrt{2}).$

T$= v/|v| = (\cos t, -\sin t)$.

$$\frac{d\mathbf{T}}{dt} = (-\sin t, -\cos t), \quad \left|\frac{d\mathbf{T}}{dt}\right| = 1.$$

$$\kappa = \frac{1}{|v|}\left|\frac{d\mathbf{T}}{dt}\right| = \cos t.$$

N$= \dfrac{\frac{d\mathbf{T}}{dt}}{|\frac{d\mathbf{T}}{dt}|} = (-\sin t, -\cos t).$

Q-2) Let C be the curve $\dfrac{x^2}{25} + \dfrac{y^2}{49} = 1$ with $x \geq 0$. Evaluate $\displaystyle\int_C \frac{xy^2}{\sqrt{(\frac{7x}{5})^2 + (\frac{5y}{7})^2}}\, ds.$

Solution:
Parametrize the curve with $\vec{r}(t) = (5\cos t, 7\sin t)$, $t \in [-\pi/2, \pi/2]$.

$v = (-5\sin t, 7\cos t)$, $|v| = \sqrt{25\sin^2 t + 49\cos^2 t}$, $ds = |v|\ dt$.

$$\frac{xy^2}{\sqrt{(\frac{7x}{5})^2 + (\frac{5y}{7})^2}} = \frac{245\cos t\sin^2 t}{\sqrt{25\sin^2 t + 49\cos^2 t}}.$$

$$\int_C \frac{xy^2}{\sqrt{(\frac{7x}{5})^2 + (\frac{5y}{7})^2}}\, ds = \int_{-\pi/2}^{\pi/2} 245\cos t\sin^2 t\ dt = \frac{490}{3}.$$

Q-3) Let $\vec{r}(t) = (\cos t, \sin t, 2t)$, $t \in [0, 2\pi]$ be a helix and F$= (x, y, xy + z)$ be a vector field on this helix. Evaluate $\displaystyle\int_{t=0}^{t=2\pi} \mathbf{F} \cdot \mathbf{T}\, ds$.

Solution:

$$\int_{t=0}^{t=2\pi} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{t=0}^{t=2\pi} \mathbf{F} \cdot d\mathbf{r} = \int_{t=0}^{t=2\pi} (\cos t, \sin t, \cos t \sin t + 2t) \cdot (-\sin t, \cos t, 2)\, dt = 8\pi^2.$$

Q-4) Find the outward flux, $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$, where $\mathbf{F}= (\arctan x, -y(1 + x^2)^{-1})$, and the curve $C = C_1 + C_2$ where C_1 is the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{49} = 1$ oriented counterclockwise and C_2 is the circle $x^2 + y^2 = 4$ oriented clockwise.

Solution:

Let $\mathbf{F}= (M, N)$. Then $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint M\, dy - N\, dx$, and by Green's theorem this last integral is equal to $\displaystyle\iint_R (M_x + N_y)\, dxdy$, where R is the region bounded by the curve C. But here $M_x = -N_y$, so the outward flux is zero.
