Math 102 Calculus – Homework 2 – Due on 21 July 2006 Friday, class time

Q-1 Evaluate the integral $\int_C x\,ds$ where C is the curve of intersection of the cylinders $x^2+y^2=1$ and $x^2+z^2=1$ in the first octant and the curve is oriented from $(1,0,0)$ towards $(0,1,1)$.

Solution: We can parameterize the curve as $x=\sqrt{1-t^2}$, $y=t$, $z=t$ with $0\le t\le 1$. Then $|v(t)|=\sqrt{2-t^2}/\sqrt{1-t^2}$ and the integral becomes $\int_0^1 \sqrt{2-t^2}\,dt = 2\int_0^{\pi/4}\cos^2\theta\,d\theta = \frac{\pi}{4}+\frac{1}{2}$.

Q-2 Let C be the unit circle in the plane traversed in the counterclockwise direction, and let $\mathbf{T}$ denote its unit tangent vector and $\mathbf{n}$ denote its unit outward normal vector. Let $f(x,y)=\ln\sqrt{x^2+y^2}$. Calculate the following integrals:

$$\int_C \nabla f\cdot\mathbf{T}\,ds \text{ and } \int_C \nabla f\cdot\mathbf{n}\,ds.$$

Solution: Parameterize the curve by $r(t)=(\cos t,\sin t)$, $0\le t\le 2\pi$.

$$\begin{aligned}\int_C \nabla f\cdot\mathbf{T}\,ds &= \int_0^{2\pi}\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right)\,dr(t)=\int_0^{2\pi}(\cos t,\sin t)\cdot(-\sin t,\cos t)\,dt=0.\\ \int_C \nabla f\cdot\mathbf{n}\,ds &= \int_0^{2\pi}\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right)\,(dy(t),-dx(t))=\int_0^{2\pi}(\cos t,\sin t)\cdot(\cos t,\sin t)\,dt\\ &= \int_0^{2\pi}dt=2\pi.\end{aligned}$$

Q-3 Show that $\omega=(y+z+yz\cos xyz)dx+(x+z+xz\cos xyz)dy+(y+x+xy\cos xyz)dz$ is exact and then evaluate the integral $\int_{(0,0,0)}^{(1,1/2,\pi)}\omega$.

Solution: Let $(M,N,P)=\omega$. Since $M_y=N_x$, $M_z=P_x$ and $N_z=P_y$, ω is exact. Then there exists a function $f(x,y,z)$ such that $\nabla f=\omega$, and the given integral has the value $f(1,1/2,\pi)-f(0,0,0)$.

Since $f_x=M$, $f=xy+xz+\sin xyz+g(y,z)$.
Since $f_y=N$, $g_y=z$ and $g(y,z)=yz+h(z)$.

Since $f_z = P$, $h'(z) = 0$ and $h(z) = c$ constant.
This gives $f = xy + yz + zx + \sin xyz + c$. Then the integral has the value $\frac{3}{2}(1+\pi)$.

Q-4 Among all simple closed smooth curves in the plane, oriented counterclockwise, find the one along which the work done by $\mathbf{F} = \left(\frac{x^2 y}{9}\right)\,\mathbf{i} + \left(x - \frac{xy^2}{25}\right)\,\mathbf{j}$ is greatest. Calculate this greatest value.

Solution: Work$=\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int\int_R \textbf{curl}\ \mathbf{F}\cdot\mathbf{k}\,dx\,dy$, where C is the curve and R is its interior. **curl F** $= N_x - M_y = 1 - x^2/9 - y^2/25$ which is nonnegative only inside the ellipse $1 = x^2/9 + y^2/25$. The curl integral will be maximal if R is the whole of the interior of this ellipse. The curl integral over the interior of this ellipse can be calculated first by making the change of variables $x = 3u$, $y = 5v$, and then passing to polar coordinates, which eventually gives $15\pi/2$ as the greatest value.

The original version of this question was wrong. Any reasonable arguments you gave for that version will receive full credits.

Q-5 Find the area of the plate cut from the plane $3x + 4y - 5z = 6$ by the planes $x = 0$, $y = 0$ and $7x + y = 3$.

Solution: Let $f = 3x + 4y - 5z - 6$. $\nabla f = (3, 4, -5)$, $p = (0, 0, 1)$. $|\nabla f| = 5\sqrt{2}$, $|\nabla f \cdot p| = 5$. Let R be the triangular region in the xy-plane bounded by the coordinate axes and the $7x + y = 3$ line.

$$\text{Area} = \int\int_R \frac{|\nabla f|}{|\nabla f \cdot p|}\,dx\,dy = \sqrt{2}\int\int_R dx\,dy = \sqrt{2}\ \text{Area}(R) = \frac{9}{7\sqrt{2}}.$$

Please send comments and questions to `sertoz@bilkent.edu.tr`