Date: 22 June 2006, Thursday Instructor: Ali Sinan Sertöz

Math 102 Calculus – Midterm Exam I – Solutions

Q-1-a) Find $\lim\limits_{n\to\infty} \dfrac{n!\, n^{1/n}}{n^n}$.

Solution: $0 \leq \dfrac{n!\, n^{1/n}}{n^n} = \dfrac{1}{n}\dfrac{2}{n}\dfrac{3}{n}\cdots\dfrac{n-1}{n}\dfrac{n}{n}\, n^{1/n} < \dfrac{1}{n}\, n^{1/n} \to 0$ as $n \to \infty$, so the limit is zero.

Q-1-b) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!\, n^{1/n}}{n^n}$ converge?

Solution: $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^{\frac{1}{n+1}}}{n^{\frac{1}{n}}} \cdot \dfrac{1}{(1+1/n)^n} \to 1/e$ as $n \to \infty$, therefore the series converges by the ratio test.

Q-2 Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n (n!)^3}{(3n)!}$ converge?

Solution: $\dfrac{a_{n+1}}{a_n} = \dfrac{3(n+1)^3}{27n^3 + \cdots} \to 1/9$ as $n \to \infty$, hence the series converges by the ratio test.

Q-3) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1+1/n}}$ converge absolutely or conditionally, or does it diverge ?

Solution: $\lim\limits_{n\to\infty} \dfrac{\frac{1}{n^{1+1/n}}}{\frac{1}{n}} = 1$, so the series of absolute values diverges by the limit comparison test.

To check the convergence of the alternating series first consider the function $f(x) = x^{1+1/x}$ for $x \geq 1$. We can easily see that $f'(x) = x^{1+1/x}\left(\dfrac{1+x-\ln x}{x^2}\right) > 0$ for $x \geq 1$, which implies that $f(x)$ is increasing and $1/f(x)$ is decreasing. Now we see that the series satisfies all the conditions of the alternating series test and converges.

Conclusion: The given series converges conditionally.

Note: Here we cannot apply p-test to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ since $1 + 1/n$ is not fixed and for any $p > 1$, we have $0 < 1 + 1/n < p$ for large n.

Q-4) Find all values of x for which the power series $\sum_{n=1}^{\infty} \frac{nx^n}{3^n(n^2+n+1)}$ converges.

Solution: $\left|\frac{a_{n+1}}{a_n}\right| \to |x|/3$ as $n \to \infty$, so the series converges absolutely for all $|x| < 3$.

When $x = 3$, the series becomes $\sum_{n=1}^{\infty} \frac{n}{(n^2+n+1)}$ and diverges by limit comparison to $\sum_{n=1}^{\infty} \frac{1}{n}$.

When $x = -3$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{(n^2+n+1)}$.

Here we first show that $\frac{n}{n^2+n+1} > \frac{n+1}{(n+1)^2+(n+1)+1}$ for $x \geq 1$.

Then we find that the series converges by the alternating series test.

Conclusion: The interval of convergence is $[-3, 3)$.

Q-5) Find $\lim_{x\to 0} \frac{6\sin x - 6x + x^3}{2x\cos x - 2x + x^3}$.

Solution: Here we use Taylor series of $\sin x$ and $\cos x$ to obtain

$$\lim_{x\to 0} \frac{6\sin x - 6x + x^3}{2x\cos x - 2x + x^3} = \lim_{x\to 0} \frac{\frac{1}{20}x^5 - \frac{1}{840}x^7 + \cdots}{\frac{1}{12}x^5 - \frac{1}{360}x^7 + \cdots} = \lim_{x\to 0} \frac{x^5\left(\frac{1}{20} - \frac{1}{840}x^2 + \cdots\right)}{x^5\left(\frac{1}{12} - \frac{1}{360}x^2 + \cdots\right)} = \frac{3}{5}.$$
