Date: 6 July 2006, Thursday NAME:...
Ali Sinan Sertöz
Time: 10:00-12:00 STUDENT NO:...

Math 102 Calculus – Midterm Exam II – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ: Check that there are 5 questions on your exam booklet. Write your name on the top of every page. Show your work in detail.

Q-1 Find $\left(\dfrac{\partial f}{\partial z}\right)_x$ at the point $(3,-2,-1)$, where $f(x,y,z) = x^2y + y^2z + z^2x$ and $x^2 + y^3 + z^5 = 0$.

Solution: We will treat z and x as free variables, and $y = y(x,z)$. Then f becomes

$$f(x,z) = x^2y(x,z) + y^2(x,z)z + z^2x.$$

Taking partial derivative of f with respect to z and keeping x fixed we get

$$\frac{\partial f}{\partial z} = x^2\frac{\partial y}{\partial z} + 2yz\frac{\partial y}{\partial z} + y^2 + 2xz. \tag{1}$$

The given constraint now takes the form

$$x^2 + y^3(x,z) + z^5 = 0.$$

Taking partial derivative of this with respect to z and keeping x fixed we get

$$3y^2\frac{\partial y}{\partial z} + 5z^4 = 0.$$

Putting $(x,y,z) = (3,-2,-1)$ we get $\dfrac{\partial y}{\partial z} = -\dfrac{5}{12}$. Substituting this value in equation (1) we get

$$\left(\frac{\partial f}{\partial z}\right)_x (3,-2,-1) = -\frac{89}{12}.$$

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Q-2 Find the minimum and maximum values of the function $f(x, y, z) = x^2 + yz - z$ on the sphere $x^2 + y^2 + z^2 = 3$.

Solution: We use Lagrange multipliers method with the constraint $g(x, y, z) = x^2 + y^2 + z^2 - 3 = 0$.
$\nabla f = (2x, z, y - 1)$, $\nabla g = 2(x, y, z)$.
$\nabla f = \lambda g \Rightarrow (2x, z, y - 1) = \lambda(x, y, z)$. Hence the system we want to solve, together with the constraint is:
(1) $2x = \lambda x$
(2) $z = \lambda y$
(3) $y - 1 = \lambda z$.
If $\lambda = 0$, then $x = z = y - 1 = 0$ and we get the point $(0, 1, 0)$ which does not satisfy the constraint $g = 0$. So $\lambda \neq 0$.

If $y = 0$, then from equation (2), we get $z = 0$. But equation (3) now gives $y = 1$, which is a contradiction. So $y \neq 0$. Since both λ and y are nonzero, equation (2) says that z is also nonzero.

Assume $x = 0$. Eliminating λ from equations (2) and (3), we get $z/y = (y - 1)/z$, or equivalently $z^2 = y^2 - y$. Adding y^2 to both sides and using $g(0, y, z) = 0$ we get $2y^2 - y - 3 = 0$ which gives $y = 3/2$ or $y = -1$.

From $g(0, 3/2, z) = 0$ we get $z = \pm\sqrt{3}/2$. And we have
$f(0, \frac{3}{2}, \frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{4} \approx 0.43$,
$f(0, \frac{3}{2}, -\frac{\sqrt{3}}{2}) = -\frac{\sqrt{3}}{4} \approx -0.43$.

From $g(0, -1, z) = 0$ we get $z = \pm\sqrt{2}$. And we have
$f(0, -1, \sqrt{2}) = -2\sqrt{2} \approx -2.82$,
$f(0, -1, -\sqrt{2}) = 2\sqrt{2} \approx 2.82$.

Next assume $x \neq 0$. Then equation (1) gives $\lambda = 2$. Putting this into equations (2) and (3), and solving for y and z we get $y = -1/3$ and $z = -2/3$. From $g(x, -1/2, -2/3) = 0$ we get $x = \pm\frac{\sqrt{22}}{3}$. And this gives
$f(\pm\frac{\sqrt{22}}{3}, -\frac{1}{3}, -\frac{2}{3}) = \frac{10}{3} \approx 3.33$.

f is continuous on a closed and bounded set, a sphere in $\mathbb{R}\diagdown^3$, so it has a global minimum and global maximum value on that sphere. Therefore those extreme values are among the values listed by this method.

We conclude that the maximum is $f(\pm\frac{\sqrt{22}}{3}, -\frac{1}{3}, -\frac{2}{3}) = \frac{10}{3}$ and the minimum is $f(0, -1, \sqrt{2}) = -2\sqrt{2}$.

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Q-3 Find local/global min/max and saddle points, if they exist, of the function $f(x, y) = x^4 + 4xy + y^4$.

Solution: $f_x = 4x^3 + 4y = 0$ and $f_y = 4y^3 + 4x = 0$ gives $(0, 0)$, $(1, -1)$ and $(-1, 1)$ as the critical points.

$\Delta = f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 - 16.$

$\Delta(0, 0) < 0$, so $(0, 0)$ is a saddle point,
$\Delta(1, -1) > 0$, and $f_{xx}(1, -1) > 0$, so $(1, -1)$ is a local minimum point,
$\Delta(-1, 1) > 0$, and $f_{xx}(-1, 1) > 0$, so $(-1, 1)$ is also a local minimum point.

The function is clearly unbounded from above, so it has no global maximum. It has a global minimum at the points $(1, -1)$ and $(-1, 1)$. At both of these points the value of the function is -2.

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Q-4 Is the following function continuous at the origin?

$$f(x,y) = \begin{cases} \dfrac{x^9 y}{x^{12} + x^6 y^2 + y^4} & \text{If } (x,y) \neq (0,0), \\ \\ 0 & \text{If } (x,y) = (0,0). \end{cases}$$

Solution: Since the degree of the denominator is larger than the degree of the numerator we suspect that it will go to zero faster than the numerator, forcing the limit to be undefined. So we try to show that the limit does not exist by trying to approach the limit along different paths.

Try $y = \lambda x^3$.

$$f(x, \lambda x^3) = \frac{\lambda x^{12}}{x^{12} + \lambda^2 x^{12} + \lambda^4 x^{12}} = \frac{\lambda}{1 + \lambda^2 + \lambda^4}$$

which shows that the limits along different paths are different.

Hence this function is not continuous at the origin and cannot be made continuous there by assigning any other value to $f(0,0)$.

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Q-5 Let $w = x^2 + xy^3 + y$, where $x = r^2 + s\cos r$, $y = 3 + \cos s$, $r = \cosh u - 1$, and $s = \pi \cos u$.

a: Using chain rule write an expression for $\dfrac{\partial w}{\partial u}$.

b: Find $\left.\dfrac{\partial w}{\partial u}\right|_{u=0}$.

Solution:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial x}{\partial s}\frac{\partial s}{\partial u}\right) + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}\frac{\partial s}{\partial u}.$$

When $u = 0$ we have $s = \pi$, $r = 0$, $x = \pi$, $y = 2$.

$\dfrac{\partial w}{\partial x} = 2x + y^3 = 2\pi + 8$ at $u = 0$,

$\dfrac{\partial x}{\partial r} = 2r - s\sin r = 0$ at $u = 0$,

$\dfrac{\partial r}{\partial u} = \sinh u = 0$ at $u = 0$,

$\dfrac{\partial x}{\partial s} = \cos r = 1$ at $u = 0$,

$\dfrac{\partial s}{\partial u} = -\pi \sin u = 0$ at $u = 0$,

$\dfrac{\partial w}{\partial y} = 3xy^2 + 1 = 12\pi + 1$ at $u = 0$,

$\dfrac{\partial y}{\partial s} = -\sin s = 0$ at $u = 0$.

Putting these together we find $\left.\dfrac{\partial w}{\partial u}\right|_{u=0} = 0.$