Date: July 18, 2007, Wednesday NAME:..
Time: 9:30-11:30
Ali Sinan Sertöz STUDENT NO:..

Math 102 Calculus II – Final Exam – Solutions

1	2	3	4	5	TOTAL
10	20	30	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.

Q-1) Check the following series for convergence:

$$\sum_{n=1}^{\infty} \frac{3 \cdot 6 \cdot 9 \cdots (3n)}{\pi^n \, n!}.$$

Solution: Use ratio test, $\dfrac{a_{n+1}}{a_n} = \dfrac{3n+3}{\pi\,(n+1)} \to \dfrac{3}{\pi} < 1$ as $n \to \infty$, to conclude that the series converges.

In fact, $a_n = \left(\dfrac{3}{\pi}\right)^n$, so the series is a geometric series which starts with $n = 1$. The sum is then found to be $\dfrac{3}{\pi}\,\dfrac{1}{1-3/\pi} = \dfrac{3}{\pi-3}$.

NAME: STUDENT NO:

Q-2) Let S be the cap cut off from the hemisphere $x^2+y^2+z^2=R^2$, $z\geq 0$ by the cylindrical surface $x^2+y^2-Ry=0$, where $R>0$. Find the surface area of S.

Solution: We first set $f=x^2+y^2+z^2-R^2$. Then $\dfrac{|\nabla f|}{|\nabla f\cdot p|}=\dfrac{R}{z}$.

$$\begin{aligned}
\text{Area}(S) &= 2\int_0^R\int_0^{\sqrt{Ry-y^2}}\frac{R}{z}\,dxdy\\
&= 2R\int_0^R\int_0^{\sqrt{Ry-y^2}}\frac{1}{\sqrt{R^2-x^2-y^2}}\,dxdy\\
&= 2R\int_0^{\pi/2}\int_0^{R\sin\theta}\frac{r\;drd\theta}{\sqrt{R^2-r^2}}\\
&= 2R\int_0^{\pi/2}\left(-\sqrt{R^2-r^2}\Big|_0^{R\sin\theta}\right)\,d\theta\\
&= 2R\int_0^{\pi/2}(R-R\cos\theta)\;d\theta\\
&= 2R^2\left(\theta-\sin\theta\Big|_0^{\pi/2}\right)\\
&= R^2(\pi-2).
\end{aligned}$$

If you choose to integrate from $\theta=0$ to $\theta=\pi$ from the beginning, then you should note that $\sqrt{\cos^2\theta}=|\cos\theta|$ which is $\cos\theta$ when $0\leq\theta\leq\pi/2$ and $-\cos\theta$ when $\pi/2\leq\theta\leq\pi$.

NAME: STUDENT NO:

Q-3) Among all rectangular regions $0 \le x \le a$, $0 \le y \le b$, find the one for which the total outward flux

$$\oint_C \mathbf{F} \cdot \vec{\mathbf{n}}\, ds$$

of $\mathbf{F} = \left(\frac{2}{3}x^3 y\right)\,\mathbf{i} + \left(2xy^3 - 17xy^2\right)\,\mathbf{j}$ across the four sides is least. Here C denotes the boundary of the given rectangle with the positive orientation.

Solution: The Green's theorem in the plane says that if $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$, then

$$\oint_C \mathbf{F} \cdot \vec{\mathbf{n}}\, ds = \int\int_{R_{ab}} (M_x + N_y)\; dxdy$$

where R_{ab} is the given rectangle with corners at the points $(0,0)$, $(a,0)$, (a,b) and $(0,b)$. We can calculate easily the above double integral;

$$\begin{aligned}\int_0^b \int_0^a \left(2x^2y + 6xy^2 - 34xy\right)\; dxdy &= \int_0^b \left(\frac{2}{3}a^3y + 3a^2y^2 - 17a^2y\right)\; dxdy \\ &= \frac{1}{3}a^3b^2 + a^2b^3 - \frac{17}{2}a^2b^2.\end{aligned}$$

We now must minimize $f(a,b) = \frac{1}{3}a^3b^2 + a^2b^3 - \frac{17}{2}a^2b^2$ where $a, b \ge 0$. On the boundary f is zero, so we look for interior critical points.

$f_a = a^2b^2 + 2ab^3 - 17ab^2 = ab^2(a + 2b - 17) = 0,$
$f_b = \frac{2}{3}a^3b + 3a^2b^2 - 17a^2b = \frac{1}{3}a^2b(2a + 9b - 51) = 0.$
This gives $(a,b) = \left(\frac{51}{5}, \frac{17}{5}\right)$ as the only interior critical point.

We calculate easily that $f(\frac{51}{5}, \frac{17}{5}) = -\frac{51^3 \cdot 17^2}{5^5 \cdot 6} < 0$. Since f is zero on the boundary and goes to infinity as a and b go to infinity, this critical point gives the global minimum.

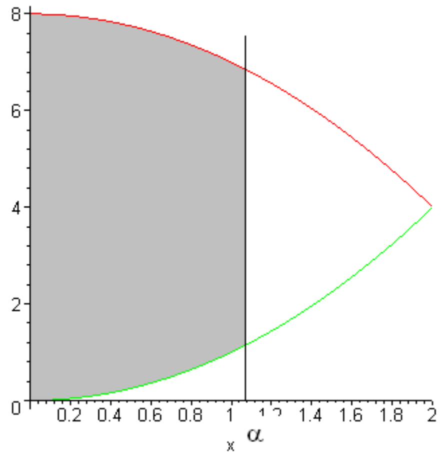
Hence the required size of the rectangle giving the minimal flux is $a = \frac{51}{5}$ and $b = \frac{17}{5}$.

Q-4) For $0 < \alpha < 2$, define

$$F(\alpha) = \int_0^{\alpha^2} \int_0^{\sqrt{y}} f\ dxdy + \int_{\alpha^2}^{8-\alpha^2} \int_0^{\alpha} f\ dxdy + \int_{8-\alpha^2}^{8} \int_0^{\sqrt{8-y}} f\ dxdy$$

where $f = \dfrac{y \sin x}{4 - x^2}\cdot$ Evaluate $F(\pi/3)$.

Solution: The region of integration is the shaded region of the following figure.



Changing the order of integration on this region we find

$$F(\alpha) = \int_0^{\alpha} \int_{x^2}^{8-x^2} \frac{\sin x}{4 - x^2}\ y\ dydx = 8 \int_0^{\alpha} \sin x\ dx = 8(1 - \cos \alpha).$$

Now we easily calculate $F(\pi/3) = 4$.

NAME: STUDENT NO:

Q-5) Let $\mathbf{F} = -y\ln(e+z^2)\ \mathbf{i} + x(x^2+y^2)\sec z\ \mathbf{j} + (x-y+z)\ln(4+x^4+y^4-z)\ \mathbf{k}$ be a field defined on the hemisphere S given by $x^2+y^2+z^2=1$, $z \geq 0$. Calculate explicitly

$$\int\int_S \text{curl } \mathbf{F} \cdot \mathbf{n}\ d\sigma.$$

Solution: We use Stokes' theorem which says

$$\int\int_S \text{curl } \mathbf{F} \cdot \mathbf{n}\ d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the boundary of S. In our case C corresponds to $z=0$, but then $\mathbf{F} = -y\ \mathbf{i} + x(x^2+y^2)\ \mathbf{j} + (x-y)\ln(4+x^4+y^4)\ \mathbf{k}$.

A parametrization for the boundary is $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + 0\ \mathbf{k}$, for $0 \leq t \leq 2\pi$. Then $d\mathbf{r}(t) = (-\sin t\ \mathbf{i} + \cos t\ \mathbf{j} + 0\ \mathbf{k})\ dt$. Putting in the parametrization of the boundary into $\mathbf{F}$ and calculating $\mathbf{F} \cdot d\mathbf{r}$ gives

$$\mathbf{F} \cdot d\mathbf{r} = (-\sin t\ \mathbf{i} + \cos t\ \mathbf{j} + \textit{something}\ \mathbf{k}) \cdot (-\sin t\ \mathbf{i} + \cos t\ \mathbf{j} + 0\ \mathbf{k})\ dt = dt.$$

Hence the right hand side integral gives 2π as the final answer.