Math 102 Calculus II – Homework II – Solutions

Due on July 13, 2007 Friday 17:00

Q-1) Let $\mathbf{F} = x^2\,\mathbf{i} + z\,\mathbf{j} + yz\,\mathbf{k}$, and C the curve parametrized as $\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t^2\,\mathbf{k}$ for $0 \le t \le \pi$. Evaluate the work integral

$$\int \mathbf{F} \cdot \mathbf{T}\, ds$$

on the curve C.

Solution:

$$\begin{aligned}
\int \mathbf{F} \cdot \mathbf{T}\, ds &= \int \mathbf{F} \cdot dr \\
&= \int (x^2, z, yz) \cdot (dx, dy, dz) \\
&= \int_0^\pi (\cos^2 t, t^2, t^2 \sin t) \cdot (-\sin t, \cos t, 2t)\, dt \\
&= \int_0^\pi \left(\sin t \cos^2 t + t^2 \cos t + 2t^3 \sin t\right)\, dt \\
&= \left[\frac{1}{3}\cos^3 t + 7t^2 \sin t - 14\,\sin t + 14\,t \cos t - 2\,t^3 \cos t \Big|_0^\pi \right] \\
&= -\frac{2}{3} + 2\pi^3 - 14\pi \approx 17.36.
\end{aligned}$$

Q-2) Find a potential function $f(x, y, z)$ for the field $\mathbf{F} = \ln x\,\mathbf{i} + \cos(y+z)\,\mathbf{j} + (z + \cos(y+z))\,\mathbf{k}$ such that $f(1, \frac{\pi}{2} - 1, 1) = -1$.

Solution: $f_x(x, y, z) = \ln x$, so $f = x \ln x - x + \phi(y, z)$.

$f_y(x, y, z) = \phi_y(y, z) = \cos(y+z)$, so $\phi(y, z) = \sin(y+z) + \alpha(z)$, and this gives $f(x, y, z) = x \ln x - x + \sin(y+z) + \alpha(z)$.

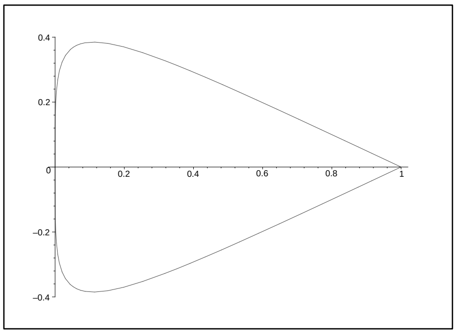
$f_z(x, y, z) = \cos(y+z) + \alpha'(z) = \cos(y+z) + z$, so $\alpha(z) = \frac{1}{2}z^2 + C$, and this gives $f(x, y, z) = x \ln x - x + \sin(y+z) + \frac{1}{2}z^2 + C$.

And finally $f(1, \frac{\pi}{2} - 1, 1) = -1$ determines C as $-\frac{3}{2}$.

Q-3) Find the area enclosed by the simple curve C parametrized as $\mathbf{r}(t) = t^4\,\mathbf{i} + (t - t^3)\,\mathbf{j}$ for $-1 \le t \le 1$. May I remind you that area is and should be a non-negative number. If you find a negative number, you owe an explanation!

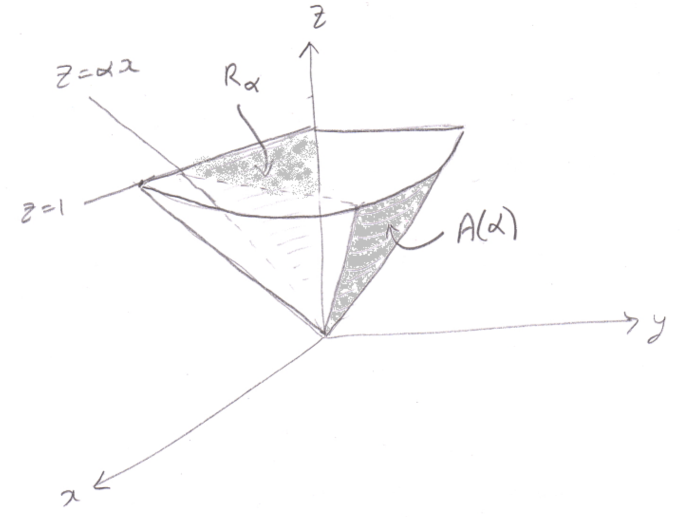
Solution: Let R be the region bounded by the curve C. Observe that the parametrization of C traverses the boundary of R in clockwise direction which is the reverse direction for the application of Green's theorem. Therefore we need the following minus sign in the formula

$$\begin{aligned}
\text{Area}(R) &= -\frac{1}{2}\int_C x\,dy - y\,dx \\
&= -\frac{1}{2}\int_{-1}^{1} [(t^4)(1-3t^2) - (t-t^3)(4t^3)]\,dt \\
&= -\frac{1}{2}\int_{-1}^{1} (-3t^4 + t^6)\,dt \\
&= -\frac{1}{2}\frac{-32}{35} = \frac{16}{35}.
\end{aligned}$$



Q-4) Let R_α be the region in the xz-plane bounded by the lines $z = \alpha x$, $z = 1$ and $x = 0$, where $\alpha \ge 1$. Let $A(\alpha)$ denote the area of the surface $z^2 = x^2 + y^2$ lying above R_α. First, without doing any calculations, find $A(1)$ and $\lim_{\alpha\to\infty} A(\alpha)$. Then calculate $A(\alpha)$ explicitly in terms of α. Check your answer with what you found above.

Solution:



$A(1)$ is the surface area of the cone lying in the first quadrant and below the plane $z = 1$, which is $\pi/(2\sqrt{2})$. When α goes to infinity, the line $z = \alpha x$ becomes the z-axis and then we have no area, giving $\lim_{\alpha \to \infty} A(\alpha) = 0$.

We now calculate the surface area of the cone over the region R_α. Here $f = x^2 + y^2 - z^2$, $\nabla f = (2x, 2y, -2z)$, $|\nabla f| = 2\sqrt{2}z$, $p = (0, 1, 0)$.

$$\begin{aligned}
A(\alpha) &= \int\int_{R_\alpha} \frac{|\nabla f|}{|\nabla f \cdot p|}\, dA \\
&= \sqrt{2} \int_0^{1/\alpha} \int_{\alpha x}^{1} \frac{z}{\sqrt{z^2 - x^2}}\, dz dx \\
&= \sqrt{2} \int_0^{1/\alpha} \left(\sqrt{z^2 - x^2} \Big|_{z=\alpha x}^{z=1} \right) dx \\
&= \sqrt{2} \int_0^{1/\alpha} (\sqrt{1 - x^2} - \sqrt{\alpha^2 - 1}\, x)\, dx \\
&= \sqrt{2} \left(\frac{x\sqrt{1 - x^2}}{2} + \frac{1}{2} \arcsin x - \frac{\sqrt{\alpha^2 - 1}}{2}\, x^2 \Big|_0^{1/\alpha} \right) \\
&= \frac{1}{\sqrt{2}} \arcsin \frac{1}{\alpha}.
\end{aligned}$$

Q-5) Let $\mathbf{F} = x \ln(1 + z^2)\, \mathbf{i} + y \tan z \cos x\, \mathbf{j} + z \ln(4 + x^4 + y^4)\, \mathbf{k}$ be a field defined on the hemisphere S given by $x^2 + y^2 + z^2 = 1$, $z \geq 0$. Calculate explicitly

$$\int\int_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

Solution: We use Stokes' theorem which says

$$\int\int_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the boundary of S. But in our case C corresponds to $z = 0$ when $\mathbf{F} = 0$, so the required integral is zero.

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sertoz@bilkent.edu.tr