Math 102 Homework-1 – Solutions
Due Date: 14 July 2008 Monday

Q-1) Let $f(x,y) = y^2 \sin\left(\frac{xy^2\pi}{6}\right)$. Evaluate the following integral:

$$\mathbf{I} = \int_{1/2}^{1}\int_{1/x}^{2} f(x,y)\,dydx + \int_{1}^{3}\int_{1}^{2} f(x,y)\,dydx + \int_{3}^{6}\int_{1}^{6/x} f(x,y)\,dydx.$$

Solution: Sketch the region, reverse the order of integration to get

$$\begin{aligned}
\mathbf{I} &= \int_1^2\int_{1/y}^{6/y} y^2 \sin\left(\frac{xy^2\pi}{6}\right)\,dxdy \\
&= \int_1^2 \left(-\frac{6\cos\left(xy^2\pi/6\right)}{\pi}\Bigg|_{1/y}^{6/y}\right) \\
&= \frac{6}{\pi}\int_1^2 \left(\cos(y\pi/6) - \cos(y\pi)\right)\,dy \\
&= \frac{6}{\pi^2}\left((6\sin(y\pi/6) - \sin(y\pi))\Big|_1^2\right) \\
&= \frac{18(\sqrt{3}-1)}{\pi^2} \approx 1.33.
\end{aligned}$$

Q-2) Find the area, in the first quadrant, that is both inside the circle $r = \sqrt{2}$ and the lemniscate $r = \sqrt{4\cos 2\theta}$.

Solution: These curves intersect when $\theta = \pi/6$. From $\theta = 0$ to $\theta = \pi/6$, it is the area of the circle, and from $\theta = \pi/6$ to $\theta = \pi/4$, it is the area of the lemniscate which constitute the common area. Hence

$$\text{Area} = \frac{\pi}{6} + \int_{\pi/6}^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,drd\theta = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2} \approx 0.657.$$

Q-3) Set up an integral to evaluate the volume of the region common to two right circular cylinders, of radii a and b where $a > b > 0$, intersecting orthogonally along their central axes.

Solution: Assume that the cylinders are $x^2 + y^2 = a^2$ and $y^2 + z^2 = b^2$. Then one eighth of the volume is given by the integral

$$\int_0^b\int_0^{\sqrt{b^2-y^2}}\int_0^{\sqrt{a^2-y^2}} dxdzdy = \int_0^b \sqrt{(a^2-y^2)(b^2-y^2)}\,dy.$$

Q-4) Find the volume of the region bounded from above by $x^2 + y^2 + z^2 = 4$, from below by $z = 1$, and from the sides by $x^2 + y^2 - 2y = 0$.

Solution: If you plot the region carefully, you will see that the volume is given by

$$2\int_0^{3/2}\int_0^{\sqrt{2y-y^2}}\int_1^{\sqrt{4-x^2-y^2}} dzdxdy + 2\int_{3/2}^{\sqrt{3}}\int_0^{\sqrt{3-y^2}}\int_1^{\sqrt{4-x^2-y^2}} dzdxdy.$$

After changing to cylindrical coordinates and evaluating these integrals you will find that their values are

$$2\left(\frac{5\pi}{9} - \frac{3\sqrt{3}}{4}\right) + 2\left(\frac{5\pi}{36}\right) = \frac{25\pi - 27\sqrt{3}}{18} \approx 1.765.$$

Q-5) Evaluate the integral

$$\int\int_R \sin^2\left(\frac{x+y}{x-y}\right) dA$$

where R is the convex quadrilateral region with vertices at the points $(1,0)$, $(2,0)$, $(0,-2)$, $(0,-1)$.

Solution: First apply the change of coordinates with $u = x + y$ and $v = x - y$. Then the integral becomes

$$\begin{aligned}\int\int_R \sin^2\left(\frac{x+y}{x-y}\right) dA &= \frac{1}{2}\int_1^2\int_{-v}^{v} \sin^2\left(\frac{u}{v}\right) dudv \\ &= \frac{1}{2}\int_1^2\int_{-v}^{v}\left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2u}{v}\right)\right) dudv \\ &= \frac{3}{4} - \frac{3}{8}\sin 2 \approx 0.409.\end{aligned}$$

Please send comments and questions to `sertoz@bilkent.edu.tr`
