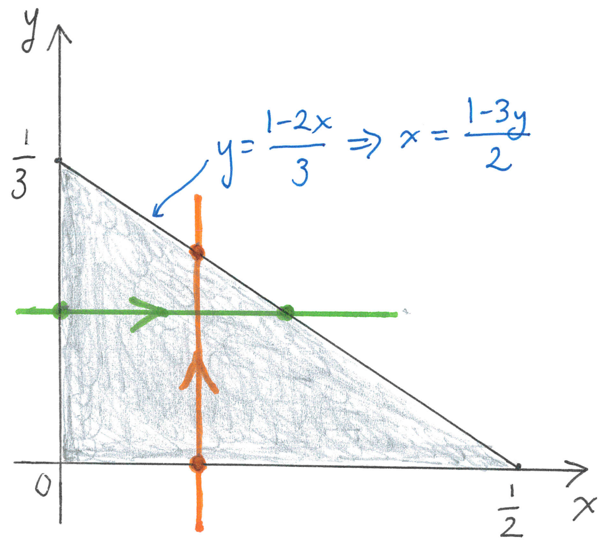
1a. Evaluate the iterated integral $\displaystyle\int_0^{1/2}\int_0^{(1-2x)/3} \sin(\pi x/(1-3y))\,dy\,dx$.

$$\int_0^{1/2}\int_0^{(1-2x)/3} \sin\left(\frac{\pi x}{1-3y}\right) dy\,dx = \iint_D \sin\left(\frac{\pi x}{1-3y}\right) dA = \int_0^{1/3}\int_0^{(1-3y)/2} \sin\left(\frac{\pi x}{1-3y}\right) dx\,dy$$

$$= \int_0^{1/3}\left[-\cos\left(\frac{\pi x}{1-3y}\right)\cdot\frac{1-3y}{\pi}\right]_{x=0}^{x=\frac{1-3y}{2}} dy = \int_0^{1/3}\frac{1-3y}{\pi}\,dy = \frac{1}{\pi}\left[y-\frac{3}{2}y^2\right]_0^{1/3} = \frac{1}{6\pi}$$

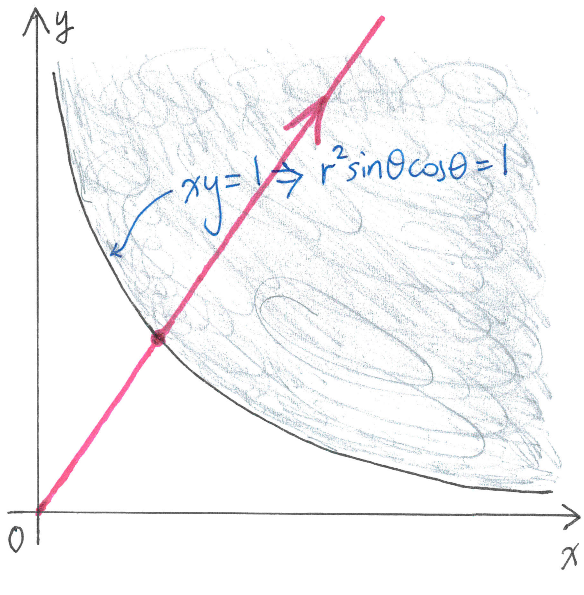


1b. Evaluate the double integral $\displaystyle\iint_D \frac{1}{(x^2+y^2)^2}\,dA$ where $D = \{(x,y) : xy \geq 1 \text{ and } x > 0\}$.

$$\iint_D \frac{1}{(x^2+y^2)^2}\,dA = \int_0^{\pi/2}\int_{\sqrt{2/\sin 2\theta}}^{\infty} \frac{1}{(r^2)^2}\cdot r\,dr\,d\theta = \int_0^{\pi/2}\left(\lim_{c\to\infty}\int_{\sqrt{2/\sin 2\theta}}^{c} r^{-3}\,dr\right) d\theta$$

$$= \int_0^{\pi/2}\lim_{c\to\infty}\left[-\frac{1}{2}r^{-2}\right]_{r=\sqrt{2/\sin 2\theta}}^{c} d\theta = \int_0^{\pi/2}\lim_{c\to\infty}\left(-\frac{1}{2}c^{-2}+\frac{1}{4}\sin 2\theta\right) d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\sin 2\theta\,d\theta = \frac{1}{4}\left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2} = \frac{1}{4}$$



2. Let V be the volume of the solid bounded by the cylinder $x^2 + y^2 = 1$ on the sides, the plane $z = y$ at the top, and the xy-plane at the bottom.

a. Only three of ❶-❹ will be graded. Mark the ones you want to be graded by putting a ✗ in the corresponding ☐s.

❶ ☒ Express V in terms of iterated integrals in Cartesian coordinates by filling in the rectangles.

$$V = \int_{\boxed{-1}}^{\boxed{1}} \int_{\boxed{0}}^{\boxed{\sqrt{1-x^2}}} \int_{\boxed{0}}^{\boxed{y}} dz\, dy\, dx$$

❷ ☒ Express V in terms of iterated integrals in Cartesian coordinates by filling in the rectangles.

$$V = \int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{z}}^{\boxed{1}} \int_{\boxed{-\sqrt{1-y^2}}}^{\boxed{\sqrt{1-y^2}}} dx\, dy\, dz$$

❸ ☒ Express V in terms of iterated integrals in cylindrical coordinates by filling in the rectangles.

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{0}}^{\boxed{r\sin\theta}} r\, dz\, dr\, d\theta$$

❹ ☐ Express V in terms of iterated integrals in spherical coordinates by filling in the rectangles.

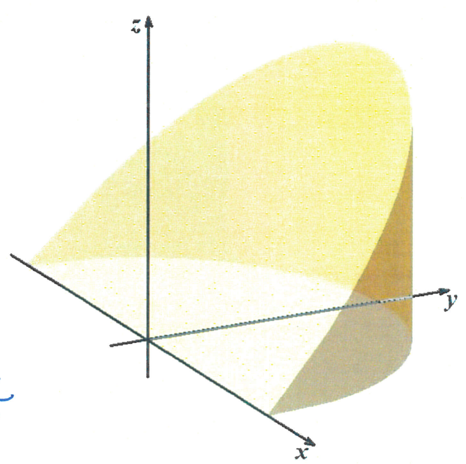
$$V = \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

b. Compute V.

$$V = \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} \int_{0}^{y} dz\, dy\, dx = \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} z \Big]_{z=0}^{z=y} dy\, dx$$

$$= \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} y\, dy\, dx = \int_{-1}^{1} \frac{1}{2} y^2 \Big]_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1-x^2)\, dx = \frac{1}{2} \left[x - \frac{1}{3} x^3 \right]_{-1}^{1} = \frac{1}{2} \left(1 - \frac{1}{3} - (-1) + \frac{1}{3} \cdot (-1) \right) = \frac{2}{3}$$



[3+3+3+3+3+3+3+3+1 *points*]

3. In each of the following, indicate all possible completions of the sentence that will make it into a true statement by putting a ✗ in the corresponding ☐s.

a. $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots\right\}$ is

☒ a convergent sequence ☐ a divergent sequence

☐ a convergent series ☐ a divergent series ☐ none of these

b. $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} + \cdots$ is

☐ a convergent sequence ☐ a divergent sequence

☒ a convergent series ☐ a divergent series ☐ none of these

c. $\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots + 1 + \cdots$ is

☐ a convergent sequence ☐ a divergent sequence

☐ a convergent series ☒ a divergent series ☐ none of these

d. $\{(-1)^{n-1}\}_{n=1}^{\infty} = \{1, -1, 1, -1, \cdots, (-1)^{n-1}, \cdots\}$ is

☐ a convergent sequence ☒ a divergent sequence

☐ a convergent series ☐ a divergent series ☐ none of these

e. $\sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + \cdots + (-1)^{n-1} + \cdots$ is

☐ a convergent sequence ☐ a divergent sequence

☐ a convergent series ☒ a divergent series ☐ none of these

f. $\{1\}_{n=1}^{\infty} = \{1, 1, 1, 1, \cdots, 1, \cdots\}$ is

☒ a convergent sequence ☐ a divergent sequence

☐ a convergent series ☐ a divergent series ☐ none of these

g. $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$ is

☐ a convergent sequence ☐ a divergent sequence

☐ a convergent series ☒ a divergent series ☐ none of these

h. $\left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots, \frac{1}{2^{n-1}}, \cdots\right\}$ is

☒ a convergent sequence ☐ a divergent sequence

☐ a convergent series ☐ a divergent series ☐ none of these

i. Mark only "none of these" in this part.

☐ a convergent sequence ☐ a divergent sequence

☐ a convergent series ☐ a divergent series ☒ none of these

4. A sequence $\{a_n\}_{n=1}^{\infty}$ satisfies

$$a_1 = 1\,,\quad a_2 = A\,,\quad \text{and} \quad a_n = \frac{a_{n-1} + a_{n-2}}{a_{n-1} - a_{n-2}} \cdot a_{n-1} \quad \text{for } n \geq 3\,,$$

where A is a real number such that $A \neq 1$, $A \neq 0$, $A \neq -1$.

a. In the following, fill in the $\boxed{}$s with real numbers that will make the sentence into a true statement.

If $A = \boxed{2}$, then $a_3 = \boxed{6}$ and $a_4 = \boxed{12}$.

b. In each of the following, fill in the $\boxed{}$ with a real number that will make the corresponding sentence into a true statement.

☐ ❶ If $A = \boxed{2}$, then $\lim_{n\to\infty} a_n = \infty$.

☐ ❷ If $A = \boxed{-2}$, then $\lim_{n\to\infty} a_n = 0$.

☒ ❸ If $A = \boxed{\sqrt{2}-1}$, then $\lim_{n\to\infty} a_n \neq 0$ and $\lim_{n\to\infty} |a_n| \neq \infty$.

c. Now choose exactly one of the statements you made in **Part b** by putting a ✗ in the corresponding ☐, and prove it fully and carefully by using correct mathematical reasoning and notation.

If $A = \sqrt{2}-1$, then $a_3 = \frac{\sqrt{2}-1+1}{\sqrt{2}-1-1} \cdot (\sqrt{2}-1) = -1$, $a_4 = \frac{-1+\sqrt{2}-1}{-1-(\sqrt{2}-1)} \cdot (-1) = 1-\sqrt{2}$,

$a_5 = \frac{1-\sqrt{2}+(-1)}{1-\sqrt{2}-(-1)} \cdot (1-\sqrt{2}) = 1$, $a_6 = \frac{1+1-\sqrt{2}}{1-(1-\sqrt{2})} \cdot 1 = \sqrt{2}-1$.

Hence the pattern $1, \sqrt{2}-1, -1, 1-\sqrt{2}, 1, \sqrt{2}-1, \ldots$ repeats.

Therefore $\lim_{n\to\infty} a_n \neq 0$ and $\lim_{n\to\infty} |a_n| \neq \infty$.