Bilkent University

Quiz # 05
Math 102-Section **10** Calculus II
14 March 2019, Thursday
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Solution Key

Q-1) Let $f(x,y) = x^4 + 2y^2 - xy$.

(i) Find the critical points of f.

(ii) Determine if each critical point is a local minimum, local maximum or a saddle point.

(iii) Does f have a global maximum?

(iv) Does f have a global minimum?

Grading: (i) 5 points, (ii) 3 points, (iii) 1 point, (iv) 1 point.

Solution:

(i)

$$f_x = 4x^3 - y = 0, \quad f_y = 4y - x = 0,$$

so the critical points are

$$p_1 = (0,0), \quad p_2(1/4, 1/16), \quad p_3 = (-1/4, -1/16).$$

(ii)

$$\begin{aligned}
&f_{xx} = 12x^2, \quad f_{yy} = 4, \quad f_{xy} = -1, \\
&\Delta(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 48x^2 - 1, \\
&\Delta(p_1) < 0, \quad p_1 \text{ is a saddle point} \\
&\Delta(p_2) > 0, \ f_{xx}(p_2 > 0, \quad p_2 \text{ is a local minimum point} \\
&\Delta(p_3) > 0, \ f_{xx}(p_3 > 0, \quad p_3 \text{ is a local minimum point}
\end{aligned}$$

(iii) No. Since $\lim\limits_{x\to\infty} f(x,0) = \infty$, the function has no global maximum.

(iv) Yes. When $|x|$ and $|y|$ are very large, f behaves like $x^4 + y^2$, so the function does not go to $-\infty$ and hence is bounded from below. Then one of the local minimum points must be the global minimum. In fact from symmetry both local minimum points give the global minimum value of f.

Here is another way to see that f is bounded from below. First observe that

$$f(x,y) = x^4 + 2y^2 - xy = \left(\sqrt{2}y - \frac{x}{2\sqrt{2}}\right)^2 + x^2\left(x^2 - \frac{1}{8}\right).$$

Let

$$g(x,y) = \left(\sqrt{2}y - \frac{x}{2\sqrt{2}}\right)^2 \text{ and } h(x) = x^2\left(x^2 - \frac{1}{8}\right).$$

Minimum value of f will be attained when $g = 0$ and h is minimum.

$$h'(x) = 4x\left(x - \frac{1}{4}\right)\left(x + \frac{1}{4}\right) = 0 \text{ when } x = 0, \frac{1}{4}, -\frac{1}{4}.$$

Then

$$h(0) = 0, \quad h(\pm\frac{1}{4}) = -\frac{1}{256}.$$

Also

$$g(\pm\frac{1}{4}, y) = 0 \text{ when } y = \pm\frac{1}{16}.$$

Hence finally

$$f(\pm\frac{1}{4}, \pm\frac{1}{16}) = -\frac{1}{256}$$

is the absolute minimum of f.

Here is the graph of $h(x) = x^2\left(x^2 - \frac{1}{8}\right)$.