Bilkent University

Quiz # 06
Math 102 - Calculus II - Section 03
24 March 2022 Thursday
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Solution Key

Q-1)

(a) Calculate, if it exists, the limit: $\lim_{(x,y)\to(0,0)} \dfrac{x^2+3xy+y}{3x^2+5y^2}$.

(b) Find all integers $k>0$ such that the following limit exists, and then find the limit.

$$\lim_{(x,y)\to(0,0)} \frac{y^2 \sin^k x}{x^4+y^4}.$$

Grading: 3+7 points

Solutions:

(a) Approach the origin along the y-axis. Then

$$\left.\frac{x^2+3xy+y}{3x^2+5y^2}\right|_{x=0} = \frac{y}{5y^2} = \frac{1}{5y},$$

and the limit as $y\to 0$ does not exit. Hence the limit does not exist by One Path Test.

(b)

$$\frac{y^2\sin^k x}{x^4+y^4} = \frac{y^2x^k}{x^4+y^4}\left(\frac{\sin x}{x}\right)^k.$$

$\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ for all positive integers k.

On the other hand

$$\lim_{(x,y)\to(0,0)} \frac{y^2x^k}{x^4+y^4} = 0 \text{ if and only if } \frac{k}{4}+\frac{2}{4} > 1.$$

This forces $k>2$. Here we used the Sertöz Theorem. 😀

Alternatively you can argue as follows. If the limit exists, then it exists and is the same along every path. In particular let $y=x$. Then

$$\left.\frac{y^2x^k}{x^4+y^4}\right|_{y=x} = \frac{x^{2+k}}{2x^4} = \frac{x^k}{2x^2},$$

and this limit exists (as $x\to 0$) if and only if $k>2$.

Now if $k=2+m$ for some positive integer m, we have

$$\frac{y^2x^k}{x^4+y^4} = \frac{x^2y^2}{x^4+y^4}\, x^m,$$

and using polar coordinates

$$0 < \left| \frac{x^2y^2}{x^4+y^4}\; x^m \right| = \left| \frac{\cos^2\theta \sin^2\theta}{\cos^4\theta + \sin^4\theta} \right| \; |x| \leq \alpha |x| \to 0 \;\text{ as }\; (x,y) \to (0,0),$$

where α is an upper bound for the function $\frac{\cos^2\theta \sin^2\theta}{\cos^4\theta+\sin^4\theta}$. Note that since the denominator is bounded away from zero, this function is bounded.

Hence the limit exists if and only if $k > 2$, and in that case the limit is zero.

For your information, here is a graph of the function $\frac{\cos^2\theta \sin^2\theta}{\cos^4\theta+\sin^4\theta}$ showing that $\alpha = 1/2$ works.