Bilkent University

Quiz # 03
Math 102 Section 09 Calculus II
26 February 2024 Monday
Instructor: Ali Sinan Sertöz
Solution Key

Q-1) (a) Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

(b) Determine if the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ converges or diverges.

(c) Find all values of x for which the series $\displaystyle\sum_{n=3}^{\infty} \frac{(x-3)^n}{\ln n}$ converges.

Show your work in detail. Correct answers with no justification will not get any credit.

Grading: 2+2+6=10 points

Solution: (Grader: `melis.gezer@bilkent.edu.tr`)

(a) $a_n = f(n)$, where $f(x) = \dfrac{\ln x}{x}$, and $f'(x) = \dfrac{1-\ln x}{x^2} < 0$, for $x \geq 3$.

Hence we can use the Integral Test.

$$\int_3^{\infty} \frac{\ln x}{x}\, dx = \left(\left. \frac{(\ln x)^2}{2} \right|_3^{\infty} \right) = \infty.$$

Thus our series diverges by the Integral Test.

(b) Let $a_n = \dfrac{1}{n^{1+\frac{1}{n}}}$ and $b_n = \dfrac{1}{n}$. We then have $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^{1+\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = 1.$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, our series also diverges by the Limit Comparison Test.

(c) We first use the Ratio Test to check for absolute convergence. Let $a_n = \dfrac{(x-3)^n}{\ln n}$. For absolute convergence we need

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} |x-3| = |x-3| < 1.$$

This gives $x \in (2, 4)$ for absolute convergence. We need to check the end points separately.

When $x = 2$, the series becomes $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{\ln n}$ and converges by the Alternating Series Test.

When $x = 3$, the series becomes $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\ln n}$, and the series diverges by Direct Comparison with the harmonic series since $\dfrac{1}{\ln n} > \dfrac{1}{n}$. Thus our series converges only for $x \in [2, 4)$.