Bilkent University

Quiz # 07
Math 102 Section 09 Calculus II
25 March 2024 Monday
Instructor: Ali Sinan Sertöz
Solution Key

Q-1) Using the Squeeze Theorem, find $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^3y^2z^5}{x^2+y^4+z^6}$.

Q-2) Find $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy^2z^3}{x^2+y^8+z^{12}}$.

Q-3) Find $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^3y^4+x^4y^3}{(x^2+y^2)^3}$.

Show your work in detail. Correct answers with no justification will not get any credit.

Grading: 3+3+4=10 points

Solution: (Grader: `melis.gezer@bilkent.edu.tr`)

(1) Notice that we have $0 \leq \left|\dfrac{x^3y^2z^5}{x^2+y^4+z^6}\right| = \dfrac{x^2}{x^2+y^4+z^6}\,|xy^2z^5| \leq |xy^2z^5|$, since the fraction $\dfrac{x^2}{x^2+y^4+z^6} \leq 1$. Now since $|xy^2z^5| \to 0$ as $(x,y,z) \to (0,0,0)$, we see that the limit $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^3y^2z^5}{x^2+y^4+z^6} = 0$ by the Squeeze Theorem.

(2) Here we have $\dfrac{1}{2}+\dfrac{2}{8}+\dfrac{3}{12} = 1$, and hence the limit does not exist by Sertöz Theorem! ☺

(3) Here put $x = r\cos\theta$ and $y = r\sin\theta$. Then we have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^3y^4+x^4y^3}{(x^2+y^2)^3} = \lim_{r\to 0} \frac{r^7(\cos^3\theta\sin^4\theta+\cos^4\theta\sin^3\theta)}{r^6(\cos^2\theta+\sin^2\theta)} = \lim_{r\to 0} r(\cos^3\theta\sin^4\theta+\cos^4\theta\sin^3\theta) = 0$$

since $(\cos^3\theta\sin^4\theta+\cos^4\theta\sin^3\theta)$ is bounded, i.e. $0 \leq |r|\,|\cos^3\theta\sin^4\theta+\cos^4\theta\sin^3\theta| \leq 2|r|$. Hence this last limit is zero by the Squeeze Theorem.

Two alternate solutions on next page.

An alternate solution for (2)

Let $f(x,y,z) = \dfrac{xy^2z^3}{x^2+y^8+z^{12}}$, and consider the path $(x,y,z) = (\lambda t^{12},\ \lambda t^3,\ \lambda t^2)$, where $t \in \mathbb{R}$ is the free parameter and λ is a non-zero constant which is not the root of the equation $1 + X^6 + X^{10} = 0$. Then moving to the origin along this path we have

$$f(\lambda t^{12},\ \lambda t^3,\ \lambda t^2) = \frac{\lambda^4}{1+\lambda^6+\lambda^{10}}.$$

We now see that along each such path f has a different limit. Therefore the limit does not exist by the Two Path Test.

An alternate solution for (3)

Observe that

$$0 \le \left|\frac{x^3y^4}{(x^2+y^2)^3}\right| = \frac{x^2}{x^2+y^2}\cdot\frac{y^2}{x^2+y^2}\cdot\frac{y^2}{x^2+y^2}\,|x| \le |x|,$$

and similarly

$$0 \le \left|\frac{x^4y^3}{(x^2+y^2)^3}\right| = \frac{x^2}{x^2+y^2}\cdot\frac{x^2}{x^2+y^2}\cdot\frac{y^2}{x^2+y^2}\,|y| \le |y|,$$

hence

$$0 \le \left|\frac{x^3y^4+x^4y^3}{(x^2+y^2)^3}\right| \le \left|\frac{x^3y^4}{(x^2+y^2)^3}\right| + \left|\frac{x^4y^3}{(x^2+y^2)^3}\right| \le |x|+|y|.$$

Thus the limit is zero as $(x,y) \to (0,0)$ by the Squeeze Theorem.