Exercise 3.20-5, page 155 Apostol: If n is a positive integer, use Theorem 3.16 to show that

$$\int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \sin(t^2)\,dt = \frac{(-1)^n}{c}, \quad \text{where} \quad \sqrt{n\pi} \le c \le \sqrt{(n+1)\pi}.$$

Lemma Let $f : [a^2, b^2] \longrightarrow \mathbb{R}$ be an integrable, non-negative, function, where $0 \le a < b$. Furthermore assume that $tf(t^2)$ is monotonic on $[a, b]$. Then

$$\int_a^b t\,f(t^2)\,dt = \frac{1}{2}\int_{a^2}^{b^2} f(x)\,dx.$$

Proof: Let $P = \{x_0, \ldots, x_r\}$ be a partition of $[a^2, b^2]$. Then $P' = \{\sqrt{x_0}, \ldots, \sqrt{x_r}\}$ is a partition of $[a, b]$.

Let $m_k = \min_{x_{k-1} \le x \le x_k} f(x)$ and $M_k = \max_{x_{k-1} \le x \le x_k} f(x)$.

Let L_r', U_r' be the lower and upper Riemann sums respectively for $tf(t^2)$ with respect to the partition P'. Similarly define L_r, U_r as the lower and upper sums for $f(x)$ with respect to the partition P. Since these functions are integrable on their respective domains we have

$$\begin{aligned} \lim_{r\to\infty} L_r' &= \lim_{r\to\infty} U_r' = \int_a^b t\,f(t^2)\,dt, \\ \lim_{r\to\infty} L_r &= \lim_{r\to\infty} U_r = \int_{a^2}^{b^2} f(x)\,dx. \end{aligned}$$

Note before we proceed that the Geometric-Arithmetic Mean theorem,

$$\sqrt{uv} \le \frac{1}{2}(u+v),$$

implies that for any $k = 1, \ldots, r$,

$$\begin{aligned} \sqrt{x_k x_{k-1}} - x_{k-1} &\le \frac{1}{2}(x_k - x_{k-1}), \\ x_k - \sqrt{x_k x_{k-1}} &\ge \frac{1}{2}(x_k - x_{k-1}). \end{aligned}$$

Assume $tf(t^2)$ is increasing. Then we have

$$\begin{aligned} L_r' &= \sum_{k=1}^r \sqrt{x_{k-1}} f(x_{k-1})\left(\sqrt{x_k} - \sqrt{x_{k-1}}\right) \\ &= \sum_{k=1}^r f(x_{k-1})\left(\sqrt{x_k x_{k-1}} - x_{k-1}\right) \\ &\le \frac{1}{2}\sum_{k=1}^r f(x_{k-1})\,(x_k - x_{k-1}) \\ &\le \frac{1}{2}\sum_{k=1}^r M_k\,(x_k - x_{k-1}) \\ &= U_r. \end{aligned}$$

Similarly we have

$$\begin{aligned}
U'_r &= \sum_{k=1}^{r} \sqrt{x_k} f(x_k) \left(\sqrt{x_k} - \sqrt{x_{k-1}}\right) \\
&= \sum_{k=1}^{r} f(x_k) \left(x_k - \sqrt{x_k x_{k-1}}\right) \\
&\geq \frac{1}{2} \sum_{k=1}^{r} f(x_k) \left(x_k - x_{k-1}\right) \\
&\geq \frac{1}{2} \sum_{k=1}^{r} m_k \left(x_k - x_{k-1}\right) \\
&= L_r.
\end{aligned}$$

We thus established the inequalities

$$L'_r \leq U_r, \quad U'_r \geq L_r.$$

Taking the limits of all sides as $r \to \infty$ we get

$$\int_a^b t\, f(t^2)\, dt \leq \frac{1}{2} \int_{a^2}^{b^2} f(x)\, dx, \quad \int_a^b t\, f(t^2)\, dt \geq \frac{1}{2} \int_{a^2}^{b^2} f(x)\, dx,$$

which establish the required equality.

If $t f(t^2)$ is decreasing, a similar argument gives the inequalities

$$L'_r \geq L_r, \quad U'_r \leq U_r,$$

which again imply the claim of the lemma.

□

Remark: If f is non-positive, then the result of the lemma holds for the non-negative function $g = -f$, and cancelling signs we have the result for f.

Remark: If $a < 0$, then a careful chase of signs gives a proof of the lemma along similar lines, so the condition on the signs of a and b can be dropped.

Theorem Let $f : [a^2, b^2] \longrightarrow \mathbb{R}$ be an integrable function. Assume that there is a finite set of points $a = x_0 < x_1 < \cdots < x_r = b$ such that $t f(t^2)$ is nonzero and monotonic on each (x_{k-1}, x_k). Then

$$\int_a^b t\, f(t^2)\, dt = \frac{1}{2} \int_{a^2}^{b^2} f(x)\, dx.$$

□

Remark: If f is continuous and changes its trend, up or down, finitely may times on the interval $[a^2, b^2]$, then $t f(t^2)$ satisfies the conditions of the theorem.

Example 1:

$$\int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} t \sin t^2\, dt = \frac{1}{2} \int_{n\pi}^{(n+1)\pi} \sin x\, dx = (-1)^n.$$

Example 2:

$$\begin{aligned}
\int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \sin t^2\,dt &= \int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \frac{t\sin t^2}{t}\,dt \\
&= \frac{1}{c}\int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} t\sin t^2\,dt \quad \text{(Theorem3.16)} \\
&= \frac{(-1)^n}{c}, \quad (\text{Example}-1)
\end{aligned}$$

for some c satisfying $\sqrt{n\pi} \le c \le \sqrt{(n+1)\pi}$. And this solves Exercise 3.19-5 on page 155 of Apostol's Calculus.

Comments to: `sertoz@bilkent.edu.tr`