Oct 13, 2003

MATH 113 HOMEWORK 1
SOLUTION MANUAL

1. Show by induction that $133 \mid 11^{n+2} + 12^{2n+1}$ for all integers $n \geq 0$.

Solution: Letting $f(n) = 11^{n+2} + 12^{2n+1}$, we want to prove that 133 divides $f(n)$ for all $n \geq 0$. We use induction. For $n = 0$, we have $f(0) = 133$, so the claim is true. We now assume that $133|f(n)$ and try to see if 133 divides $f(n+1)$. For this we follow the following sequence of equations:

$$\begin{aligned} f(n+1) &= 11^{n+3} + 12^{2n+3} \\ &= 11 \cdot 11^{n+2} + 144 \cdot 12^{2n+1} \\ &= 11 \cdot 11^{n+2} + 11 \cdot 12^{2n+1} + 133 \cdot 12^{2n+1} \\ &= 11 \cdot f(n) + 133 \cdot 12^{2n+1}. \end{aligned}$$

Since by the induction assumption $133|f(n)$, we see from the last line above that $133|f(n+1)$. This completes the induction argument and we now have the fact that $133|f(n)$ for all $n \geq 0$.

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2. Definition. If a and b are two unequal positive integers we define $\max(a, b)$ to be the larger of a and b. If $a = b$, we set $\max(a, b) = a = b$ (Thus $\max(3, 5) = \max(5, 3) = 5$, and $\max(4, 4) = 4$).

Let A_n be the following statement: If a and b are any two positive integers such that $\max(a, b) = n$, then $a = b$.

Consider the following:
Theorem. A_n is true for every integer $n \geq 1$.
Proof. a) Show A_1 is true. Let a and b be two positive integers such that $\max(a, b) = 1$. Then $a \leq 1$ and $b \leq 1$. Since they are also positive integers we must have $a = b = 1$.
b) Assume that the statement is true for some positive integer $k \geq 1$, and prove it for $k+1$. Let a and b be any two positive integers such that $\max(a, b) = k+1$. Let $\alpha = a-1$, $\beta = b-1$. Then $\max(\alpha, \beta) = k$. Since the statement A_k is true, we have that $\alpha = \beta$, so $a = b$.
Corollary. $3 = 5$.
Proof. $\max(3, 5) = 5$. Since A_5 is true, we have that $3 = 5$.

Question. What is the fallacy in the above proof?

Solution: The argument breaks down when we reduce the $k + 1$ case to the k case. When we conclude that $\max(\alpha, \beta) = k$, we are using the premise that both α and β are positive but certainly this is not true for all $a = \alpha + 1, b = \beta + 1 \geq 1$. Take for example $a = 1$, which makes $\alpha = 0$ and then clearly the premises of the statement A_k are not satisfied and consequently cannot be used.

3. Show that for all integers $n \geq 3$,

$$1^3 + 3^3 + 5^3 + \cdots + (2n+1)^3 = 18n^3 + 4n^2 - 8n - 2.$$

Solution: Let $S(n) = 1^3 + 3^3 + \cdots + (2n+1)^3$ and $P(n) = 18n^3 + 4n^2 - 8n - 2$. We want to prove that $S(n) = P(n)$ for all $n \geq 3$. We use induction. For $n = 3$ both $S(3)$ and $P(3)$ are equal to 496. Now using the assumption $S(n) = P(n)$ we try to obtain the equality $S(n+1) = P(n+1)$. No matter how hard and ingenuously we try, we fail to establish the required equality. This makes us suspect that maybe the statement is not correct after all. In fact we check that $S(4) = 1225$ and $P(4) = 1182$, so the claim is false.