Dec 15, 2003

MATH 113 HOMEWORK 5
Solutions

1-a. Page 237 Exercise 24, Evaluate $\displaystyle\int \frac{dx}{x\log x}$.

Solution: Making the substitution $u = \log x$ we get

$$\int \frac{dx}{x\log x} = \int \frac{du}{u} = \log u + C = \log|\log x| + C.$$

1-b. Page 237 Exercise 26, Evaluate $\displaystyle\int \frac{\log|x|\,dx}{x\sqrt{1+\log|x|}}$.

Solution: Making the substitution $u = \sqrt{1+\log|x|}$ we get $du = \dfrac{dx}{2x\sqrt{1+\log|x|}}$ and $\log|x| = u^2 - 1$. Then

$$\begin{aligned}\int \frac{\log|x|\,dx}{x\sqrt{1+\log|x|}} &= 2\int (u^2-1)du \\ &= \frac{2}{3}u(u^2-3) + C \\ &= \frac{2}{3}\sqrt{1+\log|x|}\,(\log|x| - 2) + C.\end{aligned}$$

2-a. Page 249 Exercise 15, Evaluate $\displaystyle\int x^2e^x dx$.

Solution: Try integration by parts with $u = x^2$ to get

$$\int x^2e^xdx = x^2e^x - 2\int xe^xdx.$$

With the second integral try integration by parts with $u = x$ to get

$$\int xe^xdx = xe^x - \int e^xdx = xe^x - e^x + C.$$

Putting these together we get

$$\int x^2e^xdx = (x^2 - 2x + 2)e^x + C.$$

2-b. Page 249 Exercise 18, Evaluate $\int x^3 e^{x^2} dx$.

Solution: First substitute $t = x^2$ to get

$$\int x^3 e^{x^2} dx = \frac{1}{2} \int te^{-t} dt.$$

Then try integration by parts with $du = t$ to get

$$\int te^{-t} dt = -te^{-t} + \int e^{-t} dt = -te^{-t} - e^{-t} + C.$$

Putting these together we get

$$\int x^3 e^{x^2} dx = -\frac{1}{2}(x^2 + 1)e^{-x^2} + C.$$

3. Page 250 Exercise 42: If n is a positive integer and if $x > 0$, show that

$$\left(1 + \frac{x}{n}\right)^n < e^x, \qquad \text{and that} \qquad e^x < \left(1 - \frac{x}{n}\right)^{-n} \quad \text{if} \quad x < n.$$

By choosing a suitable value of n, deduce that $2.5 < e < 2.99$.

Solution: For the first inequality we observe, after taking the logarithm of both sides, that the inequality holds if and only if the function $f(x) = x - n\log(1 + \frac{x}{n})$ is positive for all $x > 0$ and for all positive integers n.

We calculate that $f(0) = 0$ and $f'(x) > 0$, so $f(x) > 0$ as required, hence the first inequality holds.

Similarly, the second inequality holds if and only if the function $f(x) = x + n(1 - \frac{x}{n})$ is negative for all positive integers n and for all x with $0 < x < n$.

Again we calculate to find that under the given conditions $f(0) = 0$ and $f'(x) < 0$ so $f(x) < 0$ as required, and the second inequality holds.

Using $n = 6$ we get $2.521626376 < e < 2.985984001$.