Math 113 Homework 2 - Solutions

Due: 1 November 2005 Tuesday.

Q-1) (Page 168, Exercise 25.) Derive the following formulas:

Solution:

$$(\tan x)' = (\frac{\sin x}{\cos x})' = \frac{(\sin x)'(\cos x) - (\sin x)(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$$(\cot x)' = (\frac{\cos x}{\sin x})' = \frac{(\cos x)'(\sin x) - (\cos x)(\sin x)'}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = \csc^2 x.$$

$$(\sec x)' = (\frac{1}{\cos x})' = \frac{-(\cos x)'}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x \tan x.$$

$$(\csc x)' = (\frac{1}{\sin x})' = \frac{-(\sin x)'}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = \frac{1}{\sin x}\frac{\cos x}{\sin x} = -\csc x \cot x.$$

Q-2) (Page 173, Exercise 7) Show that the line $y = -x$ is tangent to the curve $y = x^3 - 6x^2 + 8x$.

Solution:

The slope of the tangent line to the curve at an arbitrary point x is given by $y' = 3x^2 - 12x + 8$. We find the points where this slope is -1, the slope of the line $y = -x$.

$3x^2 - 12x + 8 = -1$, $3x^2 - 12x + 9 = (3x - 3)(x - 3) = 0$, $x = 1$ and $x = 3$.

The equation of the tangent lines to the curve:
At $x = 1$, $y = 3$, tangent line is $y = -(x - 1) + 3 = -x + 4 \neq -x$.
At $x = 3$, $y = -3$, tangent line is $y = -(x - 3) - 3 = -x$.

The line $y = -x$ intersects the curve at the roots of the equation $x^3 - 6x^2 + 8x = -x$, $x^3 - 6x^2 + 9x = x(x - 3)^2 = 0$. These are the points $x = 0$ and $x = 3$. The line $y = -x$ intersects the curve at $x = 0$ and is tangent to it at $x = 3$.

Q-3) (Page 180, Exercise 17) Find $h'(x)$ and $k'(x)$ where $h(x) = f[g(x)]$ and $k(x) = g[f(x)]$.

Solution:

$h'(x) = f'[g(x)]g'(x)$. In particular $h'(0) = f'[g(0)]g'(0) = f'[2] \cdot (-5) = (2)(-5) = -10$. etc.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$h(x)$	$h'(x)$	$k(x)$	$k'(x)$
0	1	5	2	-5	0	-10	0	5
1	3	-2	0	1	1	5	1	12
2	0	2	3	1	2	4	2	-10
3	2	4	1	-6	3	12	3	4

Q-4) (Page 187, Exercise 10) Assume f has a derivative everywhere on an open interval I. Choose $a < b$ in I. Show that f' takes every value between $f'(a)$ and $f'(b)$.

Solution: Define $g : [a, b] \to \mathbb{R}$ as

$$g(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a} & \text{if } x \neq a, \\ \\ f'(a) & \text{if } x = a. \end{cases}$$

Note that $\lim_{x \to a} = g(a)$ so g is continuous throughout $[a, b]$. Then by the intermediate value property of continuous functions, g takes every value between $g(a)$ and $g(b)$. By the mean value theorem, $g(x) = f'(c)$ for some value $c \in (a, x)$. Since $g(a) = f'(a)$, every value between $g(b)$ and $f'(a)$ is taken by f'.

Similarly define $h : [a, b] \to \mathbb{R}$ as

$$h(x) = \begin{cases} \dfrac{f(x) - f(b)}{x - b} & \text{if } x \neq b, \\ \\ f'(b) & \text{if } x = b. \end{cases}$$

Using this as above we conclude that f' takes every value between $f'(b)$ and $h(a)$. We check that $h(a) = g(b)$, which concludes that f' takes every value between $f'(a)$ and $f'(b)$.

Q-5) (Page 191, Exercise 8) Plot $f(x) = \dfrac{1}{(x-1)(x-3)}$.

Solution:

$$f(x) = \frac{1}{(x-1)(x-3)}, \quad f'(x) = -2\frac{x-2}{(x-1)^2(x-3)^2}, \quad f''(x) = \frac{2(3x^2 - 12x + 13)}{(x-1)^3(x-3)^3}.$$

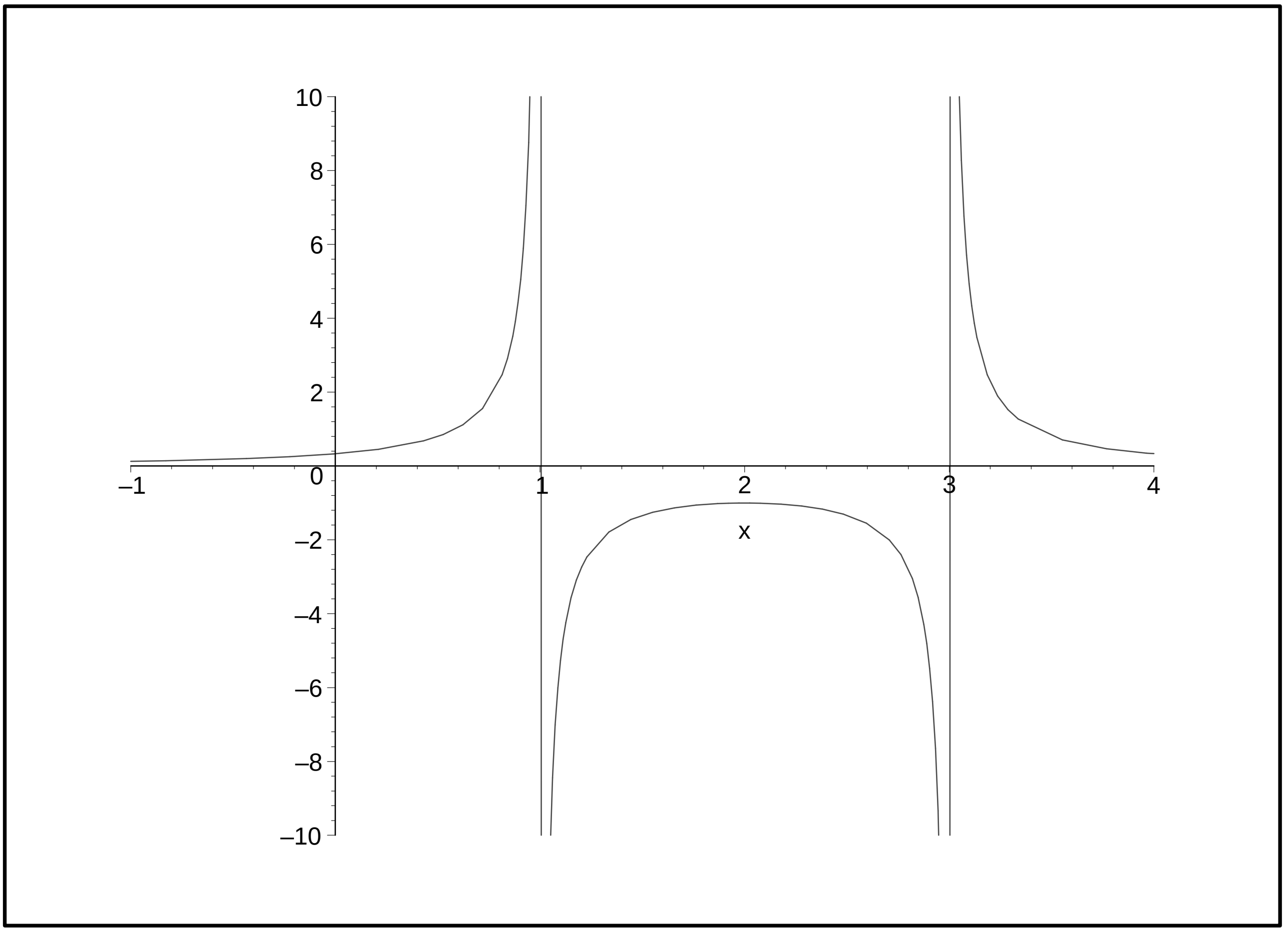
$$\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 0, \quad \lim_{x \to 1-} f(x) = \lim_{x \to 3+} f(x) = \infty, \quad \lim_{x \to 1+} f(x) = \lim_{x \to 3-} f(x) = -\infty.$$

$f'(x) > 0$ for $x < 2$. $f'(2) = 0$, and $f'(x) < 0$ for $x > 2$.

Also note that $f''(x) > 0$ for $x < 1$ and for $x > 3$, while $f''(x) < 0$ for $1 < x < 3$.

Clearly the graph has vertical asymptotes at $x = 1$ and $x = 3$.

Here is the graph:



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