Math 113 – Homework 5 – Solutions

Due: 29 November 2005 Tuesday.

Q-1) For any positive integer n define the polynomial $p_n(x) = (x-1)(x-2)\cdots(x-n)$. Find a formula for $\displaystyle\int \frac{1}{p_n(x)}\, dx$, and prove your formula.

Using partial fractions technique and using the cover-up method in determining the coefficients one easily finds that

$$\int \frac{1}{p_n(x)}\, dx = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(k-1)!(n-k)!}\ \ln|x-k| + C.$$

Q-2) Evaluate $\displaystyle\int \frac{x^3+x^2+x+1}{x^5-11\,x^4+47\,x^3-97\,x^2+96\,x-36}\, dx.$

We first observe quickly that $x=1$, $x=2$ and $x=3$ are roots, so $x^5-11\,x^4+47\,x^3-97\,x^2+96\,x-36 = (x-1)(x-2)^2(x-3)^2$.

By partial fractions technique we get

$$\frac{x^3+x^2+x+1}{(x-1)(x-2)^2(x-3)^2} = \frac{1}{x-1} + \frac{32}{x-2} + \frac{15}{(x-2)^2} - \frac{33}{x-3} + \frac{20}{(x-3)^2}.$$

Integrating this we get

$$-\frac{15}{x-2} - \frac{20}{x-3} - 33\,\ln|x-3| + \ln|x-1| + 32\,\ln|x-2| + C.$$

Q-3) Evaluate $\displaystyle\int \frac{22x^2+60x+63}{(x-3)(x^2+3x+3)^2}\, dx.$

By partial fractions technique we get

$$\frac{22x^2+60x+63}{(x-3)(x^2+3x+3)^2} = \frac{1}{x-3} - \frac{x+6}{x^2+3x+3} + \frac{x}{(x^2+3x+3)^2}.$$

Integrating this we get

$$-\frac{x+2}{x^2+3\,x+3} - \frac{11}{\sqrt{3}}\arctan\left(\frac{2\,x+3}{\sqrt{3}}\right) + \ln|x-3| - \frac{1}{2}\ln\left(x^2+3\,x+3\right) + C.$$

How to integrate: $\displaystyle\int \frac{x\,dx}{(x^2+3x+3)^2}$**:**

$$
\begin{aligned}
\int \frac{x\,dx}{(x^2+3x+3)^2} &= \frac{1}{2}\int \frac{2x+3}{(x^2+3x+3)^2}\,dx - \frac{3}{2}\int \frac{dx}{(x^2+3x+3)^2} \\
&= \frac{1}{2}\frac{-1}{x^2+3x+3} - \frac{3}{2}\int \frac{dx}{\left((x+\frac{3}{2})^2+\frac{3}{4}\right)^2} \\
&= \frac{1}{2}\frac{-1}{x^2+3x+3} - \frac{8}{3}\int \frac{dx}{\left(\frac{4}{3}(x+\frac{3}{2})^2+1\right)^2} \\
&= \frac{1}{2}\frac{-1}{x^2+3x+3} - \frac{4}{\sqrt{3}}\int \frac{dX}{(X^2+1)^2} \quad \text{where} \quad X = \frac{2}{\sqrt{3}}\left(x+\frac{3}{2}\right).
\end{aligned}
$$

How to integrate $\displaystyle\int \frac{dX}{(X^2+1)^2}$ **:**

Begin with $\displaystyle\int \frac{dX}{X^2+1}$ and use integration by parts with $u = \dfrac{1}{X^2+1}$ and $dv = dX$ to obtain

$$
\begin{aligned}
\int \frac{dX}{X^2+1} &= \frac{X}{X^2+1} + 2\int \frac{X^2}{(X^2+1)^2}\,dX \\
&= \frac{X}{X^2+1} + 2\int \frac{X^2+1-1}{(X^2+1)^2}\,dX \\
&= \frac{X}{X^2+1} + 2\int \frac{dX}{X^2+1} - 2\int \frac{dX}{(X^2+1)^2}
\end{aligned}
$$

from where we obtain

$$
\int \frac{dX}{(X^2+1)^2} = \frac{1}{2}\frac{X}{X^2+1} + \frac{1}{2}\int \frac{dX}{X^2+1}
$$

and finally

$$
\int \frac{dX}{(X^2+1)^2} = \frac{1}{2}\frac{X}{X^2+1} + \frac{1}{2}\arctan X + C.
$$

Q-4) Find $\displaystyle\int \frac{1}{\sqrt{3}\sin x - \cos x}\, dx$.

Put $u = \tan(x/2)$. Then

$$\begin{aligned}
\int \frac{1}{\sqrt{3}\sin x - \cos x}\, dx &= 2\int \frac{du}{u^2 + 2\sqrt{3}u - 1} \\
&= 2\int \frac{du}{(u+\sqrt{3}+2)(u+\sqrt{3}-2)} \\
&= \frac{1}{2}\int \left(\frac{1}{u+\sqrt{3}-2} - \frac{1}{u+\sqrt{3}+2} \right) du \\
&= \frac{1}{2}\Big(\ln(u+\sqrt{3}-2) - \ln(u+\sqrt{3}+2) \Big) + C \\
&= \frac{1}{2}\Big(\ln(\tan(x/2)+\sqrt{3}-2) - \ln(\tan(x/2)+\sqrt{3}+2) \Big) + C
\end{aligned}$$

Q-5) Find $\displaystyle\int \frac{x}{\sqrt{x^2+2x+2}}\, dx$.

First observe that

$$\begin{aligned}
\int \frac{x}{\sqrt{x^2+2x+2}}\, dx &= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+2}}\, dx - \int \frac{1}{\sqrt{x^2+2x+2}}\, dx \\
&= \sqrt{x^2+2x+2} - \int \frac{1}{\sqrt{(x+1)^2+1}}\, dx
\end{aligned}$$

For the second integral put $x+1 = \tan\theta$. The integral then becomes $\int \sec\theta\, d\theta$ which is equal to $\ln|\sec\theta + \tan\theta| + C$. Putting back the substitution and simplifying we get

$$\int \frac{x}{\sqrt{x^2+2x+2}}\, dx = \sqrt{x^2+2x+2} - \ln\left|\sqrt{x^2+2x+2} + x + 1\right| + C.$$

`Comments and questions to sertoz@bilkent.edu.tr`