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Math 113 Calculus – Midterm Exam I – Solutions

Q-1) Calculate the following limits:

a) $$\lim_{x\to 0} \frac{x^{21}+21x+\sin x}{x\cos x + x} = \lim_{x\to 0}\left(\frac{1}{1+\cos x}\right)\left(x^{20}+21+\frac{\sin x}{x}\right) = \left(\frac{1}{2}\right)(22) = 11.$$

b) $$\lim_{\theta\to 0} \frac{\sin(\sin 7\theta)}{\sin\theta} = \lim_{\theta\to 0} \frac{\sin(\sin 7\theta)}{\sin 7\theta}\,\frac{\sin 7\theta}{7\theta}\,(7)\frac{\theta}{\sin\theta} = 7.$$

c) $$\lim_{x\to 0} \frac{\sqrt{1+3x^2+4x^4}-\sqrt{1-5x^3-6x^5}}{x^2}$$

$$= \lim_{x\to 0} \frac{\sqrt{1+3x^2+4x^4}-\sqrt{1-5x^3-6x^5}}{x^2}\,\frac{\sqrt{1+3x^2+4x^4}+\sqrt{1-5x^3-6x^5}}{\sqrt{1+3x^2+4x^4}+\sqrt{1-5x^3-6x^5}}$$

$$= \lim_{x\to 0} \frac{3x^2+9x^3+6x^5}{x^2\,(\sqrt{1+3x^2+4x^4}+\sqrt{1-5x^3-6x^5})}$$

$$= \lim_{x\to 0} \frac{3+9x+6x^3}{(\sqrt{1+3x^2+4x^4}+\sqrt{1-5x^3-6x^5})} = \frac{3}{2}.$$

d) $$\lim_{x\to 0} \frac{|\sin x|}{\sqrt{3x^2-2x^3+\sin^2 x}} = \lim_{x\to 0} \frac{|\sin x|}{|x|}\,\frac{\sqrt{x^2}}{\sqrt{3x^2-2x^3+\sin^2 x}}$$ *Here we use* $|x| = \sqrt{x^2}$.

$$= \lim_{x\to 0} \frac{|\sin x|}{|x|}\,\frac{1}{\left(\frac{3x^2-2x^3+\sin^2 x}{x^2}\right)^{1/2}}$$

$$\lim_{x\to 0} \frac{|\sin x|}{|x|}\,\frac{1}{\sqrt{3-2x+\left(\frac{\sin x}{x}\right)^2}} = \frac{1}{2}.$$

Q-2) Find the volume cut from the top of a solid sphere of radius R by a plane h distance away from the center, $0 \le h \le R$.

Solution: Required volume is obtained by revolving the circle $x^2 + y^2 = R^2$ around x-axis, from $x = h$ to $x = R$.

$$V = \pi \int_h^R (R^2 - x^2)\, dx = \frac{\pi}{3}(2R^3 - 3R^2 h + h^3).$$

Q-3) Find a reasonable approximation for the integral

$$\int_0^1 \frac{x^6}{\sqrt{1+3x^3}}\, dx,$$

and give an estimate for your error.

Solution: Use the weighted mean value theorem for integrals: If f and g are continuous on $[a, b]$ and if g does not change sign on the interval, then

$$\int_a^b f(x)g(x)\, dx = f(c) \int_a^b g(x)\, dx, \quad \text{for some } c \in [a, b].$$

In this problem take $f(x) = 1/\sqrt{1+3x^3}$ and $g(x) = x^6$. For every $c \in [0, 1]$ we have $1/2 \le f(c) \le 1$, so we get

$$\frac{1}{14} \le \int_0^1 \frac{x^6}{\sqrt{1+3x^3}}\, dx \le \frac{1}{7}.$$

We may take the midpoint of the interval $[1/14, 1/7]$ as a reasonable approximation for this integral. Then the error made cannot exceed half the length of this interval. Hence we have

$$\int_0^1 \frac{x^6}{\sqrt{1+3x^3}}\, dx \approx \frac{3}{28} = 0.107...,$$

with an error not exceeding $\frac{1}{28} = 0.035....$ In fact the actual value of this integral is 0.082... which is in the predicted interval.

Q-4) Is it possible to construct a continuous function $f : [0,1] \longrightarrow [0,1]$ with the property that for every $y_0 \in [0,1]$ there are exactly two distinct $x_1, x_2 \in [0,1]$ such that $f(x_1) = f(x_2) = y_0$?

Solution: No!. The proof is an exercise in repeated use of the intermediate value property of continuous functions.

Let $a_1 < a_2$ and $b_1 < b_2$ be those points in $[0,1]$ with $f(a_1) = f(a_2) = 0$ and $f(b_1) = f(b_2) = 1$.

Case 1: Assume $b_1, b_2 \notin [a_1, a_2]$. Without loss of generality we may assume that $b_1 < a_1 < a_2$. Let $M = \frac{1}{2}\max\{f(x) | x \in [a_1, a_2]\}$ and denote by c the point in $[a_1, a_2]$ where $f(c) = M$. Then f takes the value M at least three times; once in each of the intervals $[b_1, a_1]$, $[a_1, c]$ and $[c, a_2]$. This is a contradiction. (Observe that the case $a_1 < a_2 < b_1$ is treated exactly the same.)

Case 2: Assume that only one of b_1, b_2 is in $[a_1, a_2]$. Without loss of generality we may assume that $a_1 < b_1 < a_2 < b_2$ Take any number $0 < T < 1$. Then f takes T at least three times; once in each of the intervals $[a_1, b_1]$, $[b_1, a_2]$ and $[a_2, b_2]$. This is a contradiction. (Observe that the case $b_1 < a_1 < b_2 < a_2$ is treated exactly the same.)

If both of the points b_1, b_2 are in $[a_1, a_2]$, then this is equivalent to the first case above with the roles of a_i's and b_i's switched.

This shows that no such continuous function exists.

Q-5) Let $\phi = \dfrac{1+\sqrt{5}}{2}$. Also let $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for every integer $n > 2$.

Prove that $\phi^n = \phi F_n + F_{n-1}$ for every $n \geq 2$.

Solution: We prove this by induction.

$n = 2$ case: On the one hand we have $\phi^2 = \dfrac{3+\sqrt{5}}{2}$. On the other hand $\phi F_2 + F_1 = \phi + 1 = \dfrac{3+\sqrt{5}}{2}$. So the claimed equality for $n = 2$, i.e. $\phi^2 = \phi + 1$ holds.

Now we assume that the claimed equality holds for every $k \leq n$. We start with the right hand side of the claimed equality for the $n+1$ case and obtain the left hand side:

$$
\begin{aligned}
\phi F_{n+1} + F_n &= \phi(F_n + F_{n-1}) + (F_{n-1} + F_{n-2}), \text{ property of the } F_n'\text{s} \\
&= (\phi F_n + F_{n-1}) + (\phi F_{n-1} + F_{n-2}) \\
&= \phi^n + \phi^{n-1}, \quad k = n \text{ and } k = n-1 \text{ cases} \\
&= \phi^{n-1}(\phi + 1) \\
&= \phi^{n-1}(\phi^2), \quad k = 2 \text{ case} \\
&= \phi^{n+1}.
\end{aligned}
$$

This completes the proof.
