Exercise 28, page 270, Apostol's Calculus:

28.) A function, called the *integral logarithm* and denoted by Li, is defined as follows:

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}, \text{ if } x \geq 2.$$

This function occurs in analytic number theory where it is proved that $\mathrm{Li}(x)$ is a very good approximation to the number of primes $\leq x$. Derive the following properties of $\mathrm{Li}(x)$:

(a) $\mathrm{Li}(x) = \dfrac{x}{\log x} + \displaystyle\int_2^x \frac{dt}{\log t} - \frac{2}{\log 2}.$

Solution-(a) Use integration by parts with $u = 1/\log t$ to obtain the result. □

(b) $\mathrm{Li}(x) = \dfrac{x}{\log x} + \displaystyle\sum_{k=1}^{n-1} \frac{k!x}{\log^{k+1} x} + n! \int_2^x \frac{dt}{\log^{n+1} t} + C_n,$
where C_n is a constant (depending on n). Find this constant.

Solution-(b) Assume the expression for n with $C_n = -2\displaystyle\sum_{k=0}^{n-1} \frac{k!}{\log^{k+1} 2}$. Clearly the claim holds for $n = 0$ as we showed in part (a). Take the integral $\int_2^x \frac{dt}{\log^{n+1} t}$ and apply by-parts on it with $u = 1/\log^{n+1}$ to obtain

$$\int_2^x \frac{dt}{\log^{n+1} t} = \frac{x}{\log^{n+1} x} - \frac{2}{\log^{n+1} 2} + (n+1) \int_2^x \frac{dt}{\log^{n+2} t}.$$

Putting this back into its place we see that the claim holds for $n+1$ if it holds for n, and this completes the proof by induction. □

(c) Show that there is a constant b such that $\displaystyle\int_b^{\log x} \frac{e^t}{t}\, dt = \mathrm{Li}(x)$ and find the value of b.

Solution-(c) Use the substitution $t = \log u$ to get

$$\int_b^{\log x} \frac{e^t}{t}\, dt = \int_{e^b}^x \frac{du}{\log u},$$

and this is $\mathrm{Li}(x)$ if $e^b = 2$, or if $b = \log 2$. □

(d) Express $\displaystyle\int_c^x \frac{e^{2t}}{(t-1)}\, dt$ in terms of the integral logarithm, where $c = 1 - \frac{1}{2}\log 2$.

Solution-(d) Use the substitution $t = \frac{1}{2}u + 1$ to obtain

$$\int_c^x \frac{e^{2t}}{(t-1)}\, dt = e^2 \int_{\log 2}^{2x-2} \frac{e^u}{u}\, du.$$

Now using part (c), this is equal to $e^2\mathrm{Li}(e^{2x-2})$. □

(e) Let $f(x) = e^4 \mathrm{Li}(e^{2x-4}) - e^2 \mathrm{Li}(e^{2x-2})$ if $x > 3$. Show that

$$f'(x) = \frac{e^{2x}}{x^2 - 3x + 2}.$$

Solution-(e) Using chain rule and the fundamental theorem of calculus we immediately see that $\dfrac{d\mathrm{Li}(u(x))}{dx} = \dfrac{1}{\log u(x)} \dfrac{du(x)}{dx}$. Using this we immediately get the result by direct calculation. □

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