Date: October 20, 2007, Saturday NAME:..
Time: 14:00-16:00
Ali Sinan Sertöz STUDENT NO:..

Math 113 Calculus – Midterm Exam I

1	2	3	4	5	*Bonus*	TOTAL
20	20	20	20	20	*(20)*	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5+1 questions on your exam booklet. Write your name on the top of every page. Show your work in reasonable detail but do not exaggerate. A correct answer without proper reasoning may not get any credit. Similarly a unnecessarily long explanation may be taken as an insult to intelligence and may not receive full credit. Moderation is the key word!

Q-1) Define a function on $\mathbb{R}$ as

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is continuous at every $x \in \mathbb{R}$.

Solution: Since x, $1/x$ and sine functions are continuous at every nonzero x, f itself being a combination of these is continuous when $x \neq 0$.

Since for $x \neq 0$ we have $-x \leq f(x) \leq x$, and by the sandwich theorem $\lim_{x \to 0} f(x) = 0 = f(0)$, the function is continuous also at $x = 0$.

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Q-2) Find all positive integers n such that $n^n > (n+2)!$.

Hint: First show that if the relation holds for some n then it also holds for $n+1$. Then search for the first n for which it holds. You may use the results of some homework problems if you need or remember them.

Solution: Assume $n^n > (n+2)!$. Then: $(n+1)^{n+1} = [(1+1/n)^n](1+n)n^n > [2](1+n)n^n$ where we use exercise 12b on page 45. Now the last expression is $> 2(1+n)(n+2)!$ by the induction hypothesis and this expression is $> (n+3)!$ when $n > 1$.

On the other hand check that:

$$\begin{aligned} 2^2 = 4 &\not> (2+2)! = 4! = 24, \\ 3^3 = 27 &\not> (3+2)! = 5! = 120, \\ 4^4 = 256 &\not> (4+2)! = 6! = 720, \\ 5^5 = 3125 &\not> (5+2)! = 7! = 5040, \\ 6^6 = 46656 &> (6+2)! = 8! = 40320. \end{aligned}$$

Hence the relation holds for all integers $n \geq 6$.

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Q-3) Show that

$$\frac{\pi^2}{4} \leq \int_{\pi/4}^{3\pi/4} \frac{x}{\sin x}\, dx \leq \frac{\pi^2}{2\sqrt{2}},$$

by using the weighted mean value theorem for integrals.

Solution: By the weighted mean value theorem we have

$$\int_{\pi/4}^{3\pi/4} \frac{x}{\sin x}\, dx = \frac{1}{\sin c} \int_{\pi/4}^{3\pi/4} x\, dx = \frac{1}{\sin c} \cdot \frac{\pi^2}{4}$$

for some $c \in [\pi/4, 3\pi/4]$. In this interval the minimum and the maximum values that $\sin c$ can take are $1/\sqrt{2}$ and 1, respectively. Hence the integral lies between the claimed values.

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Q-4) Let $f : [0, \infty) \to \mathbb{R}$ be a strictly increasing, bounded and continuous function. Show that f is uniformly continuous on this interval.

Solution: Let $\epsilon > 0$ be chosen.

Let $M = \sup f(x)$ for $x \in [0, \infty)$. Since f is strictly increasing, it never takes M, but there exists $x_0 \in [0, \infty)$ such that for all $x \geq x_0$, $M - f(x) < \epsilon/4$, by the definition of supremum.

For any $x, y \geq x_0$, $|f(x) - f(y)| \leq (M - f(x)) + (M - f(y)) < \epsilon/2$.

On $[0, x_0]$, f is uniformly continuous since it is continuous on a closed and bounded interval. Then there exists $\delta > 0$ such that for all $x, y \in [0, x_0]$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon/2$.

If $x < x_0 < y$ and $|x - y| < \delta$, then (i) $|f(x) - f(x_0)| < \epsilon/2$ since $x, x_0 \in [0, x_0]$ and $|x - x_0| < \delta$, and (ii) $|f(x_0) - f(y)| < \epsilon/2$ since $x_0, y \in [x_0, \infty)$. Hence $|f(x) - f(y)| < \epsilon$ by the triangle inequality.

This concludes the demonstration that f is uniformly continuous on this interval.

Here is another solution. Start with an $\epsilon > 0$. Let M be the supremum of the values of $f(x)$ for $x \in [0, \infty)$, and $m = f(0)$. Let n be the smallest positive integer satisfying $n\epsilon/2 \geq M$. Define a partition $\{y_0, \ldots, y_n\}$ of $[m, M]$ where $y_k = k\epsilon$ if $0 \leq k < n$, and $y_n = M$. Since f is strictly increasing, for every y_k with $0 \leq k < n$, there exists a unique $x_k \in [0, \infty)$ with $f(x_k) = y_k$. Let $I_k = [x_{k-1}, x_k]$ if $k = 1, \ldots, n-1$, and $I_n = [x_{n-1}, \infty)$.

For every $x, y \in I_k$ we have $|f(x) - f(y)| < y_k - y_{k-1} = \epsilon/2 < \epsilon$, where $k = 1, \ldots, n$.

Let $\delta > 0$ be the minimum of the values $x_k - x_{k-1}$ for $k = 1, \ldots, n-1$.

For any $x, y \in [0, \infty)$ with $|x - y| < \delta$ we have the following two possibilities:
(i) $x, y \in I_k$ for some $1 \leq k \leq n$.
We showed already that in this case $|f(x) - f(y)| < \epsilon$.
(ii) $x \in I_k$ and $y \in I_{k+1}$ for some $1 \leq k < n$.
Then $|f(x) - f(y)| \leq |f(x) - f(x_k)| + |f(x_k) - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon$.

This shows uniform continuity of f on $[0, \infty)$.

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Q-5) Prove or disprove: The function f defined as

$$f(x) = \begin{cases} \sin^2 \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

is integrable on $[0, 100]$.

Solution: We prove that f is integrable here.

Since f is bounded here, it remains to show that its upper and lower integrals agree.

For any $0 < \epsilon < 200$, the function is continuous and hence integrable on $[\epsilon/2, 100]$. So there exist step functions s' and t' such that $s'(x) \leq f(x) \leq t'(x)$ for all $x \in [\epsilon/2, 100]$ such that

$$0 \leq \int_{\epsilon/2}^{100} t'(x)\, dx - \int_{\epsilon/2}^{100} s'(x)\, dx < \epsilon/2.$$

Define new step functions s and t on $[0, 100]$ as

$$s(x) = \begin{cases} 0 & \text{if } 0 \leq x < \epsilon/2, \\ s'(x) & \text{if } x > \epsilon/2, \end{cases}$$

and

$$t(x) = \begin{cases} 1 & \text{if } 0 \leq x < \epsilon/2, \\ t'(x) & \text{if } x > \epsilon/2. \end{cases}$$

Then, $s(x) \leq f(x) \leq t(x)$ for all $x \in [0, 100]$ and if $\overline{I}$ and $\underline{I}$ represent the lower and upper integrals of f on this interval, we have

$$\begin{aligned} 0 \leq \overline{I} - \underline{I} &\leq \int_0^{100} t(x)\, dx - \int_0^{100} s(x)\, dx \\ &= \int_0^{\epsilon/2} 1\, dx + \int_{\epsilon/2}^{100} t'(x)\, dx - \int_0^{\epsilon/2} 0\, dx - \int_{\epsilon/2}^{100} s'(x)\, dx \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Hence f is integrable on this interval.

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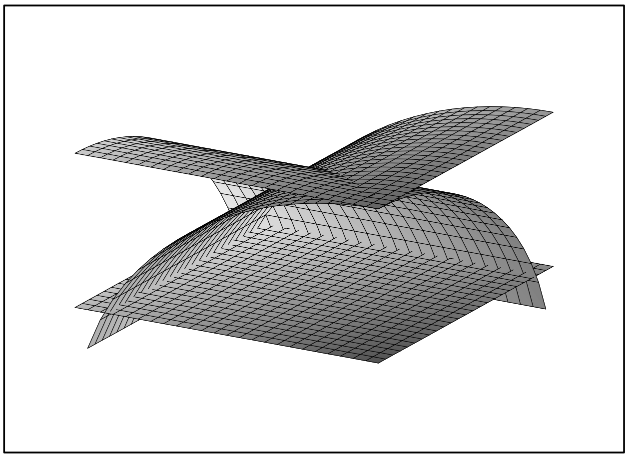
Bonus:) Find the volume of the common region of two right circular cylinders, each with base radius R, which intersect orthogonally.

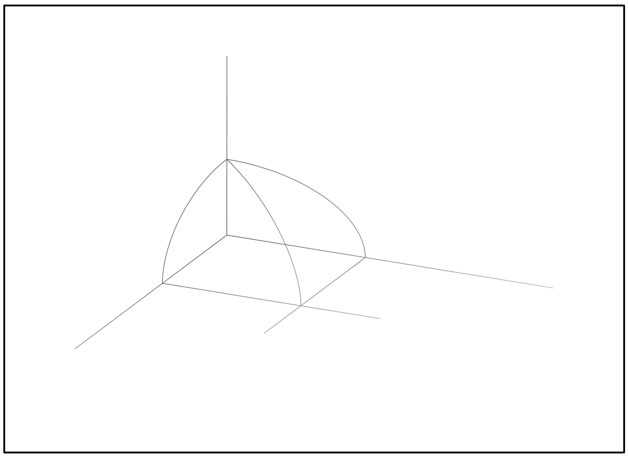
Hint: Let the cylinders be $x^2 + z^2 = R^2$ and $y^2 + z^2 = R^2$ in $\mathbb{R}^3$. Draw what is happening when $x, y, z \geq 0$.

Solution: Let S be the common part of these cylinders. If we intersect the part of S lying in the region $x, y, z \geq 0$ with a plane perpendicular to z-axis at a height of h from the xy-plane, we obtain a square of area $R^2 - h^2$. The total volume of S is 8 times the volume seen in the region $x, y, z \geq 0$. Hence

$$V(S) = 8 \int_0^R (R^2 - h^2)\, dh = \frac{16}{3} R^3.$$

The first figure shows the intersection viewed from the behind. You are actually viewing the inside of the intersection with the plane perpendicular to z-axis cutting the figure. The second figure shows the contours of the intersection, viewed from some point in the region $x, y, z \geq 0$.



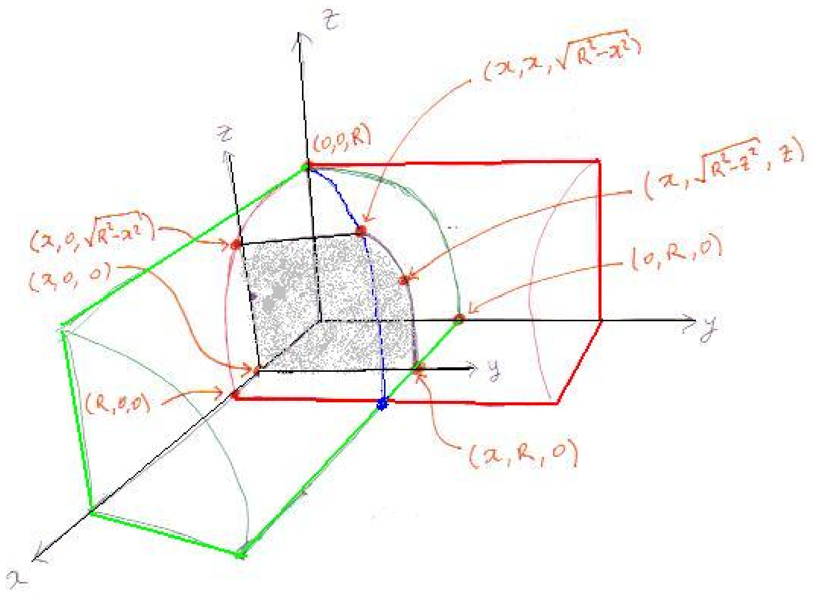


The first figure is obtained from Maple by the command:

```
plot3d({sqrt(49-x^2),sqrt(49-y^2), 3},x=0..7,y=0..7,orientation=[210,60]);
```

The second one is obtained from Maple by the command:

```
with(plots);
k:=2;m:=3/2;
spacecurve({[k*m*t,0,0],[0,k*m*t,0],
[0,0,k*m*t],[k*sin(t),k*sin(t),k*cos(t)],[k*t,k*1,0],
[k*cos(t),0,k*sin(t)],[k*1,k*t,0],[0,k*cos(t),k*sin(t)]},
t=0..Pi/2,orientation=[25,55]);
```



Here is another solution. Intersect the figure with a plane perpendicular to the x-axis at the point $(x, 0, 0)$. The result is the gray shaded region in the above figure. The area of the shaded region is the area under the function $y = \sqrt{R^2 - z^2}$ between $z = 0$ and $z = \sqrt{R^2 - x^2}$. Then to find the volume we have to integrate this area from $x = 0$ to $x = R$. Remembering that the figure above is only one-eighth of the whole figure we find that the total volume is

$$V = 8 \int_{x=0}^{x=R} \left(\int_{z=0}^{z=\sqrt{R^2-x^2}} \sqrt{R^2 - z^2}\, dz \right) dx.$$

This integral evaluates precisely to $\frac{16}{3} R^3$, but we will do that later.
