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MATH 114 Homework 1 Solutions

1. Evaluate the following improper integral:

$$\int_0^{+\infty} x^2 e^{-x}\, dx.$$

Solution. $\displaystyle\int_0^{+\infty} x^2 e^{-x}\, dx = \lim_{b\to\infty} \int_0^b x^2 e^{-x}\, dx$. By using integration by parts twice

$$\begin{aligned}
\int_0^b \underbrace{x^2}_{u} \underbrace{e^{-x}\, dx}_{dv} &= -x^2 e^{-x}\Big|_0^b + 2\int_0^b \underbrace{x}_{u} \underbrace{e^{-x}\, dx}_{dv} \\
&= -x^2 e^{-x}\Big|_0^b + 2\left(-xe^{-x}\Big|_0^b + \int_0^b e^{-x} dx\right) \\
&= -x^2 e^{-x}\Big|_0^b + 2\left(-xe^{-x}\Big|_0^b - e^{-x}\Big|_0^b\right) \\
&= -b^2 e^{-b} - 2be^{-b} - 2e^{-b} + 2.
\end{aligned}$$

So $\displaystyle\int_0^{+\infty} x^2 e^{-x}\, dx = \lim_{b\to\infty} \left(-b^2 e^{-b} - 2be^{-b} - 2e^{-b} + 2\right) = 2.$

2. For a certain value of the constant C the following improper integral converges. Determine C and evaluate the integral.

$$\int_2^{+\infty} \left(\frac{Cx}{x^2+1} - \frac{1}{2x+1}\right) dx.$$

Solution.

$$\begin{aligned}
\int_2^{+\infty} \left(\frac{Cx}{x^2+1} - \frac{1}{2x+1}\right) dx &= \lim_{b\to\infty} \int_2^b \left(\frac{Cx}{x^2+1} - \frac{1}{2x+1}\right) dx \\
&= \lim_{b\to\infty} \left(\frac{C}{2}\ln(x^2+1) - \frac{1}{2}\ln(2x+1)\right)\Bigg|_2^b \\
&= \lim_{b\to\infty} \frac{1}{2}\ln\left(\frac{(b^2+1)^C}{2b+1}\right) - \frac{1}{2}\ln\left(\frac{5^C}{5}\right).
\end{aligned}$$

Now the above limit exists if and only if $C = \frac{1}{2}$ in which case the integral is equal to $\frac{1}{4}\ln 5$.

3. a) Find the function $f(x)$ such that

$$f(x) = \sum_{n=1}^{+\infty} n^2 x^n = x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \cdots, -1 < x < 1$$

Solution. Start with

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots, -1 < x < 1: \text{ take derivative} \\
\frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots, -1 < x < 1: \text{ multiply by } x \\
\frac{x}{(1-x)^2} &= \sum_{n=1}^{\infty} n x^{n} = x + 2x^2 + 3x^3 + 4x^4 + \cdots, -1 < x < 1: \text{ take derivative} \\
\frac{1+x}{(1-x)^3} &= \sum_{n=1}^{\infty} n^2 x^{n-1} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots, -1 < x < 1: \text{ multiply by } x \\
\frac{x+x^2}{(1-x)^3} &= \sum_{n=1}^{\infty} n^2 x^{n} = x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \cdots, -1 < x < 1.
\end{aligned}$$

So $f(x) = \dfrac{x+x^2}{(1-x)^3}$.

b) Find the following sum:

$$-\frac{1}{2} + \frac{2^2}{2^2} - \frac{3^2}{2^3} + \frac{4^2}{2^4} - \frac{5^2}{2^5} + \cdots$$

Solution. This sum is $f(-1/2)$. Since $-1 < -1/2 < 1$, we have that

$$-\frac{1}{2} + \frac{2^2}{2^2} - \frac{3^2}{2^3} + \frac{4^2}{2^4} - \frac{5^2}{2^5} + \cdots = f\left(-\frac{1}{2}\right) = \frac{-\frac{1}{2} + \left(-\frac{1}{2}\right)^2}{\left(1 - \left(-\frac{1}{2}\right)\right)^3} = -\frac{2}{27}.$$

4. Use the identity $\cos^2 x = (1 + \cos 2x)/2$ to obtain the Maclaurin series for $\cos^2 x$. Then differentiate this series to obtain the Maclaurin series for $-2\sin x \cos x$. Check that this is the series for $-\sin 2x$.

Solution. Start with

$$\cos t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots + (-1)^n \frac{t^{2n}}{(2n)!} + \cdots, \text{ all } t.$$

Putting $t = 2x$, adding 1 and dividing by 2, we get

$$\cos^2 x = 1 - \frac{2^1 x^2}{2!} + \frac{2^3 x^4}{4!} - \cdots + (-1)^n \frac{2^{2n-1} x^{2n}}{(2n)!} + \cdots = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n-1} x^{2n}}{(2n)!}, \text{ all } t.$$

Now take derivative of both sides.

$$\begin{aligned}
-2\cos x \sin x &= -\frac{2^1}{2!} 2x + \frac{2^3}{4!} 4x^3 - \cdots + (-1)^n \frac{2^{2n-1}}{(2n)!} 2n x^{2n-1} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n-1}}{(2n)!} 2n x^{2n-1} \\
&= -\left(2x - \frac{(2x)^3}{3!} + \cdots + (-1)^{n-1} \frac{(2x)^{2n-1}}{(2n-1)!} + \cdots\right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(2x)^{2n-1}}{(2n-1)!} \\
&= -\sin(2x).
\end{aligned}$$