## MATH 114 Homework 3 Solutions

**1.** Let

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

By using $\varepsilon$-$\delta$ definition of limit, show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

**Solution.** Remember the following inequality:

$$a \geq 0, b \geq 0 \Rightarrow 2ab \leq a^2 + b^2 \qquad (*)$$

This follows from $0 \leq (a-b)^2 = a^2 + b^2 - 2ab$. Now given $\varepsilon > 0$, let $\delta = \varepsilon/2$. Let $(x,y)$ be any point such that $0 < (x-0)^2 + (y-0)^2 < \delta^2$, i.e. $0 < x^2 + y^2 < \delta^2$. Then $|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2} < \delta$. So

$$|f(x,y) - 0| = \frac{x^2|y|}{x^2+y^2} = |x|\frac{|x|\;|y|}{|x|^2+|y|^2} \overset{(**)}{\leq} |x|\frac{1}{2} < \frac{\delta}{2} = \varepsilon$$

where the inequality $(**)$ follows from $(*)$ with $a = |x|,\ b = |y|$.

**2.** Find $\lim\limits_{(x,y)\to(0,0)} \arctan\left(\dfrac{x-y}{x^2+y^2}\right)$ or show that the limit doesn't exist.

**Solution.** By using the two path test we show that the limit doesn't exist.
First consider the path $y = x$ through the point $(0,0)$. Then

$$\lim_{\substack{(x,y)\to(0,0) \\ y=x}} \arctan\left(\frac{x-y}{x^2+y^2}\right) = \lim_{x\to 0} \arctan \frac{0}{2x^2} = \lim_{x\to 0} \arctan 0 = 0.$$

Next consider the path $y = 0$.

$$\lim_{\substack{(x,y)\to(0,0) \\ y=0}} \arctan\left(\frac{x-y}{x^2+y^2}\right) = \lim_{x\to 0} \arctan \frac{1}{x} = \begin{cases} +\infty & \text{if } x \to 0^+ \\ -\infty & \text{if } x \to 0^- \end{cases}$$

As $0 \neq \mp\infty$, we have that "limit doesn't exist".

**3.** For the following functions verify that $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$.

a) $f(x,y) = xy^2 + x^2y^3 + x^3y^4$
b) $f(x,y) = x^y$, $x > 0$.

**Solution.**
a) $\dfrac{\partial f}{\partial x} = y^2 + 2xy^3 + 3x^2y^4 \Rightarrow \dfrac{\partial^2 f}{\partial y \partial x} = 2y + 6xy^2 + 12x^2y^3$ and
$\dfrac{\partial f}{\partial y} = 2xy + 3x^2y^2 + 4x^3y^3 \Rightarrow \dfrac{\partial^2 f}{\partial x \partial y} = 2y + 6xy^2 + 12x^2y^3$. So $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$.

b) $\dfrac{\partial f}{\partial x} = yx^{y-1} \Rightarrow \dfrac{\partial^2 f}{\partial y \partial x} = x^{y-1} + yx^{y-1}\ln x$ and
$\dfrac{\partial f}{\partial y} = x^y \ln x \Rightarrow \dfrac{\partial^2 f}{\partial x \partial y} = yx^{y-1}\ln x + x^y \dfrac{1}{x} = yx^{y-1}\ln x + x^{y-1}$. So $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$.

**4.** Let $f(u, v)$ be a differentiable function such that

$$\frac{\partial f}{\partial u}(2,3) = -1, \quad \frac{\partial f}{\partial u}(12,1) = -3, \quad \frac{\partial f}{\partial v}(2,3) = 4, \quad \frac{\partial f}{\partial v}(12,1) = 2.$$

Let $z = f(x^2y, x^2 - y)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(x, y) = (2, 3)$.

**Solution.** Let $u = x^2y,\ v = x^2 - y$. By the Chain Rule:

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}(u,v)\,\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}(u,v)\,\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}(u,v)\,2xy + \frac{\partial f}{\partial v}(u,v)\,2x.$$

Now substitute $x = 2,\ y = 3$. Note that then $u = 12,\ v = 1$. So

$$\frac{\partial z}{\partial x}(2,3) = \underbrace{\frac{\partial f}{\partial u}(12,1)}_{-3} \cdot 12 + \underbrace{\frac{\partial f}{\partial v}(12,1)}_{2} \cdot 4 = -36 + 8 = -28.$$

Similarly

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}(u,v)\,\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}(u,v)\,\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}(u,v)\,x^2 + \frac{\partial f}{\partial v}(u,v)\,(-1).$$

So

$$\frac{\partial z}{\partial y}(2,3) = \underbrace{\frac{\partial f}{\partial u}(12,1)}_{-3} \cdot 4 + \underbrace{\frac{\partial f}{\partial v}(12,1)}_{2} \cdot (-1) = -12 - 2 = -14.$$