Date: 22 May 2006, Monday
Instructor: Ali Sinan Sertöz
Time: 9:00-11:00

Math 114 Calculus – Final Exam – Solutions

Q-1) Consider the power series $\displaystyle\sum_{n=2}^{\infty} \frac{2^n(x-1)^n}{n \ln n}$.

(i): Find its radius of convergence. (6 points)

(ii): Check convergence at the end points. (7 points each)

Solution: Applying the ratio test $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2|x-1| < 1$ for convergence gives the radius of convergence as $1/2$.

When $x = 1/2$, the series converges by alternating series test.

When $x = 3/2$, the series diverges by integral test.

Q-2) Find all local/global min/max and saddle points, if any, of the function

$$f(x, y) = 4x^2 - 6xy + 5y^2 - 20x + 26y.$$

Justify your answers.

Solution: $f_x = 8x - 6y - 20 = 0$ and $f_y = -6x + 10y + 26 = 0$ gives $p_0 = (1, -2)$ as the only critical point. Since f must be bounded below, we can immediately conclude at this stage that this is the global minimum. Applying the second derivative test with $f_{xx} = 8$, $f_{xy} = -6$ and $f_{yy} = 10$ gives $\Delta > 0$, also indicating a minimum.

Q-3) The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the point on this ellipse closest to the origin.

Solution: This is a solved example in the book, page 1046. You apply Lagrange multiplier method to the function $x^2 + y^2 + z^2$ with the constraints $x + y + z = 1$ and $x^2 + y^2 = 1$, to get both $(1, 0, 0)$ and $(0, 1, 0)$ as the points on the ellipse closest to the origin.

Q-4) Evaluate $\displaystyle\int_0^2 \int_y^2 x^2 \sin(xy)\, dxdy$.

Solution:

$$\int_0^2 \int_y^2 x^2 \sin(xy)\, dxdy = \int_0^2 \int_0^x x^2 \sin(xy)\, dydx = -\int_0^2 x \left(\cos(xy)|_0^x \right)\, dx$$

$$= -\int_0^x (x\cos(x^2) - x)\, dx = \left(-\frac{1}{2}\sin x^2 + \frac{1}{2}x^2 \Big|_0^2 \right) = -\frac{1}{2}\sin 4 + 2.$$

Q-5) Let C be the circle of intersection of the plane $x-2y+3z=0$ with the sphere $x^2+y^2+z^2=13$, oriented counterclockwise when viewed from the north pole of the sphere.
Calculate $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\, ds$, where $\mathbf{F}=(x+y)\mathbf{i}+(3x-z)\mathbf{j}+(5y-7x)\mathbf{k}$ and $\mathbf{T}$ is the unit tangent vector of C with the given orientation.

Solution: Let D be the disc bounded by C with its unit normal vector $\mathbf{n}=(1,-2,3)/|(1,-2,3)|$. The area of D is 13π. The Stokes' theorem gives

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathbf{T}\, ds &= \int\int_D \mathbf{curlF}\cdot\mathbf{n}\, d\sigma \\
&= \frac{1}{\sqrt{14}}\int\int_D (6,7,2)\cdot(1,-2,3)\, d\sigma \\
&= \frac{-2}{\sqrt{14}}\int\int_D d\sigma \\
&= \frac{-2}{\sqrt{14}}\, 13\pi = \frac{-26\pi}{\sqrt{14}}.
\end{aligned}$$
