Due on February 20, 2006, Monday

MATH 114 Homework 2

Find the limits of the following sequences, if they converge.

1. $a_n = \dfrac{n!}{n^n}$.

Solution:

$$a_n = \frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdots \frac{n}{n} \leq \frac{1}{n}$$

since each $\dfrac{k}{n} \leq 1$ for $k = 2, \ldots, n$. Hence $\lim\limits_{n\to\infty} a_n = 0$ by the sandwich theorem.

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2. $a_n = \dfrac{n!}{10^{6n}}$.

Solution: Let $N = 10^6$ and $K = \dfrac{N!}{N^N}$. For $n > N$, we note that each $\dfrac{n}{N} > 1$ and

$$a_n = K \cdot \left(\frac{N+1}{N} \cdots \frac{n-1}{N} \right) \cdot \frac{n}{N} > \frac{K}{N}\, n.$$

Hence $\lim\limits_{n\to\infty} a_n = \infty$.

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3. $a_n = \left(\dfrac{1}{n}\right)^{1/\ln n}$.

Solution: $\ln a_n = -1$, so $a_n = 1/e$ for all n, giving us the trivial limit $1/e$.

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4. $a_n = \dfrac{n^2}{3n-1} \sin \dfrac{1}{n}$.

Solution: $a_n = \dfrac{n}{3n-1} \cdot \dfrac{\sin(1/n)}{(1/n)} \to \dfrac{1}{3}$ as $n \to \infty$.

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5. $a_n = \dfrac{(\ln n)^{2006}}{n}$.

Solution: After one application of L'Hopital, we find

$$\lim_{n\to\infty} \frac{(\ln n)^{2006}}{n} = \lim_{n\to\infty} \frac{2006(\ln n)^{2005}}{n}.$$

Clearly, after 2006 applications we will have

$$\lim_{n\to\infty} \frac{(\ln n)^{2006}}{n} = \cdots = \lim_{n\to\infty} \frac{2006!}{n} = 0.$$
