Due on February 27, 2006, Monday

MATH 114 Homework 3 – Solutions

1: Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{n^2 \sin(1/n)}{3n-1} x^n$.

Solution:

Let $a_n(x) = \frac{n^2 \sin(1/n)}{3n-1} x^n$. Use the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^2 \cdot \frac{3n-1}{3n+1} \right| |x| = |x|.$$

The series converges absolutely for $|x| < 1$. We now check the end points:

$\lim_{n\to\infty} |a_n(\pm 1)| = \lim_{n\to\infty} \frac{n}{3n-1} \cdot \left| \frac{\sin 1/n}{1/n} \right| = \frac{1}{3}$, so the series diverges at the end points by the divergence test. The interval of convergence is $(-1, 1)$.

2: Find the interval of convergence of the power series $f(x) = \sum_{n=1}^{\infty} n^2 x^n$.

Do not forget to check the end points. Show that $f(x)$ is a rational function of x.

Solution:

Let $a_n(x) = n^2 x^n$. Use the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} |x| = |x|.$$

The series converges absolutely for $|x| < 1$. We now check the end points:

$a_n(\pm 1)| = n^2$, so $\lim_{n\to\infty} a_n(\pm 1) \neq 0$ and the series diverges at the end points by the divergence test. The interval of convergence is $(-1, 1)$.

Observe that $f(x) = x \cdot \left(x \cdot \left(\frac{1}{1-x} \right)' \right)' = \frac{x(1+x)}{(1-x)^3}$.

3: Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{n!}{n^n}\, x^n$.

Do not forget to check the end points.

Solution:

Let $a_n(x) = \frac{n!}{n^n}\, x^n$. Use the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \lim_{n\to\infty} \frac{|x|}{(1+1/n)^n} = \frac{|x|}{e}.$$

The series converges absolutely for $|x| < e$. We now check the end points:

We first claim that $\left(1+\frac{1}{n}\right)^n < e$ for $n \geq 1$.

Let $f(x) = \left(1+\frac{1}{x}\right)^x$ for $x \geq 1$.

$$f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ln\left(1+\frac{1}{x}\right) - \frac{1}{1+x}\right].$$

We claim that $f'(x) > 0$ for $x \geq 1$.

Let $g(x) = \ln\left(1+\frac{1}{x}\right) - \frac{1}{1+x}$ for $x \geq 1$. Then $g'(x) = -\frac{1}{x(1+x)^2} < 0$ for $x \geq 1$.

We now observe that $g(1) = \ln 2 - \frac{1}{2} = 0.1931... > 0$ and $\lim_{x\to\infty} g(x) = 0$.

Therefore $g(x) > 0$ for $x \geq 1$.

Also observe that $f(1) = 2$, $\lim_{x\to\infty} f(x) = e$ and $f'(x) = \left(1+\frac{1}{x}\right)^x g(x) > 0$ for $x \geq 1$.
Therefore we have $2 < \left(1+\frac{1}{x}\right)^x < e$ for $x \geq 1$.

Now we put these together to check convergence at the end points:

$\left|\frac{a_{n+1}(\pm e)}{a_n(\pm e)}\right| = \frac{e}{\left(1+\frac{1}{n}\right)^n} > 1$ so $|a_n(\pm e)| > |a_n(\pm e)|$ for $n \geq 1$. This clearly implies that $\lim_{n\to\infty} a_n(\pm e) \neq 0$ so the series diverges at the end points by the divergence test.

The interval of convergence is $(-e, e)$.

4: Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{n!}{2^n} x^n$.
Do not forget to check the end points.

Solution:

Let $a_n(x) = \frac{n!}{2^n} x^n$. Use the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \lim_{n\to\infty} \frac{n+1}{2} |x| = \begin{cases} 0 & \text{if } x = 0, \\ \infty & \text{if } x \neq 0. \end{cases}.$$

The series converges converges only when $x = 0$.

5: Assume that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for all $x \in (-c, c)$ for some $c > 0$. Show that when f is an odd function, then $c_{2n} = 0$ for all n. Also show that when f is even, then $c_{2n+1} = 0$ for all n.

Solution:

Assume f is odd: $f(x) + f(-x) = 0$.

$$0 = \sum_{n=0}^{\infty} [1 + (-1)^n] c_n x^n = 2 \sum_{n=0}^{\infty} c_{2n} x^{2n}, \text{ so } c_{2n} = 0 \text{ for all } n.$$

Similarly assume f is even: $f(x) - f(-x) = 0$.

$$0 = \sum_{n=0}^{\infty} [1 - (-1)^n] c_n x^n = 2 \sum_{n=0}^{\infty} c_{2n+1} x^{2n+1}, \text{ so } c_{2n+1} = 0 \text{ for all } n.$$

Comments to sertoz@bilkent.edu.tr.