Calculus 114 – Homework 1
Please take your homework solutions to room SA144, Ali Adalı's office
on **February 27, 2008 Wednesday**.

Kummer's Test: Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

(1) $\sum_{n=1}^{\infty} a_n$ is convergent if and only if there exist *(i)* a sequence of positive terms $\{p_n\}$, *(ii)* a positive number $c > 0$, and *(iii)* an index N, such that

$$p_n \left(\frac{a_n}{a_{n+1}} \right) - p_{n+1} \geq c, \text{ for all } n \geq N.$$

(2) $\sum_{n=1}^{\infty} a_n$ is divergent if and only if there exist a sequence of positive terms $\{p_n\}$ such that $\sum_{n=1}^{\infty} 1/p_n$ diverges and

$$p_n \left(\frac{a_n}{a_{n+1}} \right) - p_{n+1} \leq 0, \text{ for all } n \geq N.$$

Raabe's Test: Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms.

(1) If there is an index N and a number $L > 1$ such that

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{L}{n}, \text{ for all } n \geq N,$$

then the series $\sum_{n=1}^{\infty} a_n$ converges.

(2) If $\frac{a_{n+1}}{a_n} \geq 1 - \frac{1}{n}$, for all $n \geq N$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Gauss' Test: Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms.
Suppose there exist an index N, a real number $s > 1$, and constants $M > 0$ and $L \in \mathbb{R}$ such that

$$\frac{a_{n+1}}{a_n} = 1 - \frac{L}{n} + \frac{f(n)}{n^s}, \text{ for all } n \geq N,$$

where $|f(n)| \leq M$ for all $n \geq N$.

(1) If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

(2) If $L \leq 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Optional exercise: (do not hand in your solutions to Ali for this question.)

Q-0) Prove Kummer's test. Observe that each statement is an "if and only if" statement. Using Kummer's test prove the other tests, including the ratio test. You may need to use Bernoulli's inequality at some point. Also recall the obvious fact that a series with positive terms converges if and only if its sequence of partial sums is bounded.

In the following exercises you can use any test, including the ones mentioned here.

It is highly recommended that you first solve the routine problems from Thomas's Calculus, pages 746-786. Do not consider yourself *studied* until you solve at most half of the exercises there.

Q-1) Consider the series $\sum_{n=1}^{\infty} \frac{(n!)^{\alpha}}{(3n)!}$, where α is a real constant. Find all values of α for which the series converges.

Q-2) Find all values of $\alpha \in \mathbb{R}$ for which the following series converges.

$$\sum_{n=1}^{\infty} \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right)^{\alpha}.$$

After you solve this, you should be able to give a quick answer for the same question for the following series:

$$\sum_{n=1}^{\infty} \left(\frac{1 \cdot 6 \cdot 11 \cdots (5n+1)}{3 \cdot 8 \cdot 13 \cdots (5n+3)} \right)^{\alpha}.$$

Q-3) Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}.$$

Q-4) Find all pairs $(a, b) \in \mathbb{N} \times \mathbb{N}$ for which the following series converges.

$$\sum_{n=3}^{\infty} \frac{1}{n^{a}(\ln n)^{b}}.$$

Please forward any comments or questions to `sertoz@bilkent.edu.tr`