Due Date: March 2, 2011 Wednesday NAME:..................................................

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# Math 114 Calculus – Homework 1 – Solutions

| 1 | 2 | 3 | 4 | TOTAL |
|---|---|---|---|---|
| | | | | |
| 25 | 25 | 25 | 25 | 100 |

*Please do not write anything inside the above boxes!*

Check that there are 4 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

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**Q-1)** Assume that each $a_n > 0$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$. Show that $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$.

**Solution:**

Case-1: $\rho > 0$.

Let $\epsilon > 0$ be chosen arbitrarily such that $0 < \epsilon < \rho$. Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$, there exists an index $N$ such that

$$\rho - \epsilon < \frac{a_{n+1}}{a_n} < \rho + \epsilon, \text{ for all } n \geq N. \tag{1}$$

In particular, since $a_n > 0$ for all $n$, we have

$$(\rho - \epsilon)a_N < a_{N+1} < (\rho + \epsilon)a_N.$$

We claim that

$$(\rho - \epsilon)^k a_N < a_{N+k} < (\rho + \epsilon)^k a_N \text{ for all } k \geq 1.$$

The $k = 1$ case is already done above. Assume that the claim is true for some $k \geq 1$. Then from Equation 1 we have for $n = N + k$,

$$(\rho - \epsilon)a_{N+k} < a_{N+k+1} < (\rho + \epsilon)a_{N+k}.$$

Now using the induction hypothesis on $a_{N+k}$, we get

$$(\rho - \epsilon)^{k+1} a_N < a_{N+k+1} < (\rho + \epsilon)^{k+1} a_N.$$

This is of the claimed form and this proves the claim. We showed that, for this given $\epsilon > 0$, there exists an index $N$ such that, for all $k \geq 1$ we have

$$\rho - \epsilon < \left(\frac{a_{N+k}}{a_N}\right)^{\frac{1}{k}} < \rho + \epsilon.$$

This shows that

$$\lim_{k\to\infty} \left(\frac{a_{N+k}}{a_N}\right)^{\frac{1}{k}} = \rho.$$

Since

$$\lim_{k\to\infty} (a_N)^{\frac{1}{k}} = 1 \text{ and } \lim_{k\to\infty} \left(\frac{k}{N+k}\right) = 1,$$

We have

$$\lim_{k\to\infty} (a_{N+k})^{\frac{1}{N+k}} = \lim_{k\to\infty} \left[(a_{N+k})^{\frac{1}{k}}\right]^{\frac{k}{N+k}} = \rho.$$

Or, by setting $n = N + k$, we have as required

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.$$

Case-2: $\rho = 0$.

This case is exactly similar, and in fact easier than the first case. Simply replace $(\rho - \epsilon)$ by $0$ in the above arguments and all else works fine.

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**Q-2)** Define a function $f : \mathbb{R} \to \mathbb{R}$ as follows.

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

(i) Sketch the graph of $y = f(x)$.
(ii) Show that $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$.
(iii) Show that $f$ is $C^\infty$ but is not analytic at the origin.

**Solution:**

We first attack the second part.

Observe that $f'(x) = \dfrac{2}{x^3}e^{-1/x^2}$ when $x \neq 0$. To calculate the derivative at $x = 0$, we calculate the limit

$$\lim_{x\to 0} \frac{e^{-1/x^2} - 0}{x - 0} = \lim_{t\to\infty} \frac{t}{e^{t^2}} = 0,$$

where we used L'Hopital's rule to get the last limit.

Next we claim that

$$f^{(n)}(x) = P_n(\frac{1}{x})e^{-1/x^2} \text{ for } x \neq 0, \text{ and } f^{(n)}(0) = 0,$$

for all $n \geq 1$, where $P_n(t)$ is a polynomial in $t$.

Note that the $n = 1$ case is already proved above. Assume for $n - 1$ and check for $n$.

That the form of the $n$-th derivative when $x \neq 0$ is straightforward differentiation. To calculate the $n$-th derivative of $f$ at $x = 0$ we need to calculate

$$\lim_{x\to 0} \frac{f^{(n-1)}(x) - 0}{x - 0} = \lim_{x\to 0} \frac{P_{n-1}(\frac{1}{x})e^{-1/x^2}}{x} = \lim_{t\to\infty} \frac{tP_{n-1}(t)}{e^{t^2}} = 0,$$

where we used repeatedly the L'Hopital's rule to get the last limit.

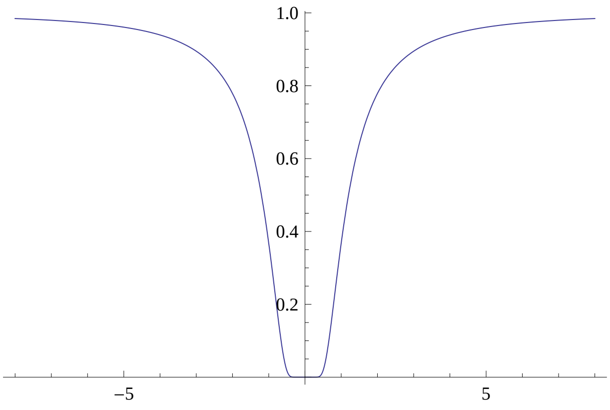
This proves the (ii) part.

For the (iii) part, observe that the Taylor series of $f$ at $x = 0$ is written as

$$f(0) + f'(0)x + \frac{f''(0) = (2!^2}{x} + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \equiv 0,$$

whereas the function is not identically equal to zero. hence the taylor series of the function at $x = 0$ does not converge to the function in any neighborhood of $x = 0$. This means the function is not analytic at the origin.

Finally we return to the first part. The derivative of the function is $f'(x) = \dfrac{2}{x^3}e^{-1/x^2}$ when $x \neq 0$. Then the function is decreasing for $x < 0$ and increasing for $x > 0$ with an absolute minimum at $x = 0$. The second derivative vanished and changes sign when $x = \pm\sqrt{2/3}$, which then gives concavity information.

Here is the graph: 

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**Q-3)** Let $f : [1, \infty) \to \mathbb{R}$ be an increasing function.

(i) Show that

$$f(1) + \cdots + f(n-1) < \int_1^n f(x)\,dx < f(2) + \cdots + f(n).$$

(ii) Choosing a suitable $f$, show that $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$.

(iii) Does the series $\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$ converge?

(iv) Does the series $\sum_{n=1}^{\infty} \frac{n^n}{e^n n!}$ converge?

**Solution:**

The **(i)** part is straightforward using Riemann sums.

**(ii)** Let $f(x) = \ln x$. Then $\int_1^n \ln x\,dx = \left( x\ln x - x\Big|_1^n \right) = n\ln n - (n-1) = \ln \frac{n^n}{e^{n-1}}$. Putting this into (i) we get

$$\ln(n-1)! < \ln \frac{n^n}{e^{n-1}} < \ln n!$$

or equivalently

$$(n-1)! < \frac{n^n}{e^{n-1}} < n!. \tag{2}$$

Putting $n+1$ for the left side inequality in this equation we obtain

$$n! < \frac{(n+1)^{n+1}}{e^n}.$$

Combining with the right inequality in Equation 2 we get

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}.$$

Dividing all sides by $n^n$ we get

$$\frac{e}{e^n} < \frac{n!}{n^n} < \frac{n+1}{e^n}\left(1 + \frac{1}{n}\right)^n. \tag{3}$$

Take the $n$-th root of all sides to obtain

$$\frac{e^{1/n}}{e} < \frac{\sqrt[n]{n!}}{n} < \frac{(n+1)^{1/n}}{e}\left(1 + \frac{1}{n}\right).$$

Taking limits of all sides as $n$ goes to infinity we find

$$\frac{1}{e} \le \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} \le \frac{1}{e},$$

which is the required result.

Here is another idea due to **Murat Can**: Let $a_n = \frac{n!}{n^n}$. Apply ratio test to check the convergence of the series $\sum a_n$.

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} \\
&= \frac{1}{(1+1/n)^n} \\
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e}.
\end{aligned}$$

From question-1 we know that the root test will also give the same limit. Hence

$$\lim_{n\to\infty} = \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

**(iii)** Set $a_n = \dfrac{e^n n!}{n^n}$.

Multiply the left side inequality of equation 3 by $e^n$to get

$$e < \frac{e^n n!}{n^n} = a_n.$$

Hence $\sum a_n$ diverges since the general term is bounded away from $0$.

**(iv)** Set $a_n = \dfrac{n^n}{e^n n!}$.

Take reciprocal of all sides of equation 3 and divide all sides by $e^n$ to get

$$\frac{1}{(n+1)(1+1/n)^n} < \frac{n^n}{e^n n!} < \frac{1}{e}.$$

Since we have $(1+1/n)^n < e$ for all $n$, we get

$$\frac{1/e}{n+1} < \frac{1}{(n+1)(1+1/n)^n} < \frac{n^n}{e^n n!} = a_n.$$

Now, the series $\sum a_n$ diverges by direct comparison with the divergent series $\sum (1/e)/(n+1)$.

A note on (iii) and (iv): Let $a_n = \dfrac{e^n n!}{n^n}$. The Stirling formula says

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1,$$

which is equivalent to writing

$$\lim_{n\to\infty} \frac{a_n}{\sqrt{2\pi n}} = 1,$$

which in turn implies that

$$\lim_{n\to\infty} a_n = \infty.$$

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**Q-4)** Find the sum $1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots + \frac{(-1)^n}{1+3n} + \cdots$.

**Solution:**

Let $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{1+3n}}{1+3n}$ for $-1 < x \leq 1$.

Then

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n (x^3)^n = \frac{1}{1+x^3} = \frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)},$$

which we can write as

$$f'(x) = \frac{1}{3}\frac{1}{x+1} - \frac{1}{6}\frac{2x-1}{x^2-x+1} + \frac{2}{3}\frac{1}{\left(\frac{2x-1}{\sqrt{3}}\right)^2 + 1}.$$

Integrating from $0$ to $x$ we find, since $f(0) = 0$,

$$f(x) = \frac{1}{3}\ln(1+x) - \frac{1}{6}\ln(x^2 - 2x + 1) + \frac{1}{\sqrt{3}} \arctan \frac{2x-1}{\sqrt{3}} + \frac{\pi}{6\sqrt{3}}.$$

Finaly, the sum we want to calculate is

$$f(1) = \frac{1}{3}\ln 2 + \frac{\pi}{3\sqrt{3}} \approx 0.83564884\ldots.$$