Due Date: March 2, 2012 Friday class time NAME:...................................................

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# Math 114 Calculus – Homework 2 – Solutions

| 1 | 2 | 3 | 4 | TOTAL |
|---|---|---|---|---|
| 25 | 25 | 25 | 25 | 100 |
| 25 | 25 | 25 | 25 | 100 |

*Please do not write anything inside the above boxes!*

Check that there are 4 questions on your booklet. Write your name on top of every page.
Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.
Everything you write on your paper should be part of a well constructed sentence. No hanging equations will be read. No sequence of equations will be read unless they are part of a well constructed, meaningful sentence.

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**Q-1)** Fix a real number $a \geq 0$. For each such number define a function

$$f_a(x) = \left(1 + \frac{1}{x}\right)^{x+a}, \text{ for } x \geq 1.$$

(i) Show that $\lim_{x\to\infty} f_a(x) = e$ for all $a \geq 0$.

(ii) Show that, when $0 \leq a < 1/2$, $f_a(x)$ is strictly increasing on the interval $(a/(1-2a), \infty)$, and in particular $f_a(x) < e$ on this interval.

(iii) Show that, when $a > 1/2$, $f_a(x)$ is strictly decreasing, and in particular $f_a(x) > e$ for all $x \geq 1$.

**Solution:** Set $g_a(x) = \ln f_a(x)$. Then

$$\begin{aligned}
\lim_{x\to\infty} g_a(x) &= \lim_{x\to\infty} (x+a)\ln\left(1+\frac{1}{x}\right) \\
&= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x+a}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{\left(1+\frac{1}{x}\right)}\frac{-1}{x^2}}{\frac{-1}{(x+a)^2}} \\
&= \lim_{x\to\infty} \frac{(x+a)^2}{x(x+1)} \\
&= 1.
\end{aligned}$$

Equivalently,

$$\lim_{x\to\infty} f_a(x) = e, \text{ for all } a \geq 0.$$

Observe that

$$g_a'(x) = \ln\left(1+\frac{1}{x}\right) - \frac{x+a}{x(x+1)}, \text{ and } \lim_{x\to\infty} g_a'(x) = 0.$$

Also note that

$$g_a''(x) = \frac{(2a-1)x+a}{x^2(x+1)^2}.$$

When $0 \le a < 1/2$, we see that $g_a''(x) < 0$ for all $x > a/(1-2a)$, hence $g_a'(x)$ is decreasing on this interval. As $g_a'(x) \downarrow 0$ as $x$ goes to infinity, $g_a'(x) > 0$ for all $x > a/(1-2a)$. This means that $g_a(x)$ is increasing on this interval, and as its limit is $1$, it is always less than $1$.

This translates as

$$\left(1+\frac{1}{x}\right)^{x+a} < e, \text{ for all } x > \frac{a}{1-2a}, \text{ when } 0 \le a < \frac{1}{2}.$$

When $a \ge 1/2$, we always have $g_a''(x) > 0$, so that $g_a'(x)$ is increasing. As $g_a'(x) \uparrow 0$ as $x$ goes to infinity, $g_a'(x) < 0$ for all $x \ge 1$. This means that $g_a(x)$ is decreasing on this interval, and as its limit is $1$, it is always greater than $1$.

This translates as

$$\left(1+\frac{1}{x}\right)^{x+a} > e, \text{ for all } x \ge 1, \text{ when } a \ge \frac{1}{2}.$$

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**Q-2)** Using the results of Question-1, show that

$$n^{1/3} < \frac{e^n n!}{n^n} < 3n, \text{ for all } n = 1, 2, 3, \ldots.$$

**Solution:** First we want to show that $\frac{e^n n!}{n^n} > n^{1/3}$ for all $n = 1, 2, \ldots$.

We showed that

$$\left(1+\frac{1}{x}\right)^{x+a} < e, \text{ for all } x > \frac{a}{1-2a}, \text{ when } 0 \le a < \frac{1}{2}.$$

When $a = 1/3$, we see that $\frac{a}{1-2a} = 1$, and in fact

$$\left(1+\frac{1}{x}\right)^{x+1/3} < e, \text{ for all } x \ge 1.$$

Now we prove the claim by induction. When $n = 1$, there is nothing to prove as $e > 1$. Assume that the claim holds for some integer $n \ge 1$, and examine the expression for $n + 1$ as follows.

$$\begin{aligned}
\frac{e^{n+1}(n+1)!}{(n+1)^{n+1}} &= \frac{e^n n!}{n^n} \cdot \frac{e}{(1+1/n)^n} \\
&> n^{1/3} \cdot \frac{e}{(1+1/n)^n} = n^{1/3} \frac{e}{(1+1/n)^n} \frac{(n+1)^{1/3}}{(n+1)^{1/3}} \\
&= (n+1)^{1/3} \frac{e}{(1+1/n)^{n+1/3}} \\
&> (n+1)^{1/3},
\end{aligned}$$

which completes the induction and the proof.

Next we want to show that $\frac{e^n n!}{n^n} < 3n$ for all $n = 1, 2, \ldots$.

Setting $a = 1$ for the result of Question-1, we see that

$$\left(1+\frac{1}{x}\right)^{x+1} > e, \text{ for all } x \ge 1.$$

Again we prove the claim by induction. When $n = 1$, there is nothing to prove as $e < 3$. Assume that the claim holds for some integer $n \ge 1$, and examine the expression for $n + 1$ as follows.

$$\begin{aligned}
\frac{e^{n+1}(n+1)!}{(n+1)^{n+1}} &= \frac{e^n n!}{n^n} \cdot \frac{e}{(1+1/n)^n} \\
&< 3n \cdot \frac{e}{(1+1/n)^n} = 3n \frac{e}{(1+1/n)^n} \frac{n+1}{n+1} \\
&= 3(n+1) \frac{e}{(1+1/n)^{n+1}} \\
&< 3(n+1),
\end{aligned}$$

which completes the induction and the proof.

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**Q-3)** Using Stirling's formula, show that there exist real numbers $0 < \alpha < \beta$ such that

$$\alpha\, n^{1/2} < \frac{e^n n!}{n^n} < \beta\, n^{1/2},$$

for all sufficiently large $n$.

**Solution:**

Stirling's formula is the equality which states that

$$\lim_{n\to\infty} \frac{e^n n!}{n^n} \cdot \frac{1}{\sqrt{2\pi}\, n^{1/2}} = 1.$$

Using the definition of limit, corresponding to $\epsilon = 1/2$, there exists an index $N$ such that for all $n \geq N$, we have

$$\frac{1}{2} < \frac{e^n n!}{n^n} \cdot \frac{1}{\sqrt{2\pi}\, n^{1/2}} < \frac{3}{2},$$

or equivalently

$$\left(\frac{1}{2}\sqrt{2\pi}\right) n^{1/2} < \frac{e^n n!}{n^n} < \left(\frac{3}{2}\sqrt{2\pi}\right) n^{1/2},$$

as claimed.

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**Q-4)** Find the radius of convergence for each of the series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}\, x^n, \text{ and } \sum_{n=1}^{\infty} \frac{n^n}{n!}\, x^n.$$

Find also the convergence behaviors of each series at the end points of its interval of convergence. You will notice that the purpose of all the previous work in this homework was to be able to examine these end points.

*(Grading for this problem: $1$ point for each radius of convergence, $8$ points for correctly examining the end points for the first series, and $15$ points for the end points of the second series.)*

**Solution:**

Let $a_n(x) = \frac{n!}{n^n}\, x^n$. Apply ratio test to find

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{|x|}{(1+1/n)^n} \to \frac{|x|}{e} \text{ as } n \to \infty.$$

The series absolutely converges for $|x| < e$, hence the radius of convergence is $e$.

From Question-2 we know that $a_n(e) > n^{1/3}$, hence $\lim_{n\to\infty} a_n(e) = \lim_{n\to\infty} |a_n(-e)| = \infty$. Thus the series diverges at both end points, by divergence test.

Therefore the interval of convergence for $\sum_{n=1}^{\infty} \frac{n!}{n^n}\, x^n$ is $(-e,\, e)$.

Now let $b_n(x) = \frac{n^n}{n!}\, x^n$. Apply ratio test to find

$$\left|\frac{b_{n+1}(x)}{b_n(x)}\right| = (1+1/n)^n |x| \to e|x| \text{ as } n \to \infty.$$

The series absolutely converges for $|x| < 1/e$, hence the radius of convergence is $1/e$.

From Question-2 we know that $b_n(1/e) > 1/(3n)$, hence $\sum_{n=1}^{\infty} \frac{n^n}{n!e^n}$ diverges by comparison with the harmonic series.

From Question-2 we also know that $a_n(1/e) < 1/(n^{1/3})$. Check also that

$$b_{n+1}(1/e) = \frac{(1+1/n)^n}{e}\, b_n(1/e) < b_n(1/e).$$

Hence $b_n(1/e) \downarrow 0$ as $n \to \infty$. Therefore, by the alternating series test, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!e^n}$ converges.

Finally, we conclude that the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}\, x^n$ is $[-1/e, 1/e)$.

Observe that the examining of the end points depends on giving convenient bounds for $a_n(e) = 1/b_n(1/e)$. Therefore we could have used the results of Question-3 in the above arguments with the same effectiveness.