Summer 2007-08 MATH 116 Homework 3 Solutions

1. Find the outward flux of the vector field

$$\mathbf{F}(x,y) = (x^2+y^2)\sin(x)\mathbf{i} + e^{xy^2}\ln(\frac{y}{e})\mathbf{j}$$

across the rectangle with vertices $(0,1)$, $(\pi,1)$, $(0,e)$, and (π,e).

Solution : Let C denote the rectangle with vertices $(0,1)$, $(\pi,1)$, $(0,e)$, and (π,e).

We can consider C as the union of the following four line segments: C_1 from $(0,1)$ to $(\pi,1)$, C_2 from $(\pi,1)$ to (π,e), C_3 from (π,e) to $(0,e)$, and C_4 from $(0,e)$ to $(0,1)$.

Assume that $\mathbf{n}$ denotes the outward-pointing unit normal vector on C.

The curve C_1 can be parameterized by $\mathbf{r}_1(t) = t\mathbf{i} + \mathbf{j}$ for t in $[0,\pi]$.
Hence on C_1 we have $\mathbf{F}\cdot\mathbf{n} = \big((t^2+1)\sin(t)\mathbf{i} - e^t\mathbf{j}\big)\cdot(-\mathbf{j}) = e^t$.

The curve C_2 can be parameterized by $\mathbf{r}_2(t) = \pi\mathbf{i} + t\mathbf{j}$ for t in $[1,e]$.
Hence on C_2 we have $\mathbf{F}\cdot\mathbf{n} = \Big(e^{\pi t^2}\ln(\frac{t}{e})\mathbf{j}\Big)\cdot(\mathbf{i}) = 0$.

The curve C_3 can be parameterized by $\mathbf{r}_3(t) = -t\mathbf{i} + e\mathbf{j}$ for t in $[-\pi,0]$.
Hence on C_3 we have $\mathbf{F}\cdot\mathbf{n} = \big(-(t^2+e^2)\sin(t)\mathbf{i}\big)\cdot(\mathbf{j}) = 0$.

The curve C_4 can be parameterized by $\mathbf{r}_4(t) = -t\mathbf{j}$ for t in $[-e,-1]$.
Hence on C_4 we have $\mathbf{F}\cdot\mathbf{n} = \big(\ln(\frac{-t}{e})\mathbf{j}\big)\cdot(-\mathbf{i}) = 0$.

Hence the outward flux of the vector field $\mathbf{F}(x,y)$ across the rectangle C can be calculated as follows:

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{C_1}\mathbf{F}\cdot\mathbf{n}\,ds + \int_{C_2}\mathbf{F}\cdot\mathbf{n}\,ds + \int_{C_3}\mathbf{F}\cdot\mathbf{n}\,ds + \int_{C_4}\mathbf{F}\cdot\mathbf{n}\,ds$$

$$= \int_0^\pi e^t\,dt + 0 + 0 + 0 = e^t\big|_0^\pi = e^\pi - 1$$

2. Find the work done by the force

$$\mathbf{F}(x,y,z) = (2xz^3 + e^y)\mathbf{i} + (xe^y + 4y^3\cos(z))\mathbf{j} + (3x^2z^2 - y^4\sin(z))\mathbf{k}$$

over the curve parameterized by

$$\mathbf{r}(t) = \sin(\pi t)\mathbf{i} + t^3\mathbf{j} + (2t-1)\mathbf{k}, \quad \text{for } 0 \le t \le \frac{1}{2}$$

in the direction of increasing t.

Solution : First we will check whether if the vector field $\mathbf{F}(x,y,z)$ is conservative on $\mathbb{R}^3$. We have

$$\begin{aligned}
\frac{\partial}{\partial y}\left(2xz^3 + e^y\right) &= e^y &&= \frac{\partial}{\partial x}\left(xe^y + 4y^3\cos(z)\right), \\
\frac{\partial}{\partial z}\left(xe^y + 4y^3\cos(z)\right) &= -4y^3\sin(z) &&= \frac{\partial}{\partial y}\left(3x^2z^2 - y^4\sin(z)\right), \\
\frac{\partial}{\partial z}\left(2xz^3 + e^y\right) &= 6xz^2 &&= \frac{\partial}{\partial x}\left(3x^2z^2 - y^4\sin(z)\right).
\end{aligned}$$

By the above three equalities we know that $\mathbf{F}(x,y,z)$ is conservative on $\mathbb{R}^3$. Now let's try to find a potential function $f(x,y,z)$ for $\mathbf{F}(x,y,z)$. We know that we should have $f_x(x,y,z) = 2xz^3 + e^y$. Hence we have

$$f(x,y,z) = x^2z^3 + xe^y + g(y,z)$$

for some function $g(y,z)$. We also know that we should have $f_y(x,y,z) = xe^y + 4y^3\cos(z)$. This means $g_y(y,z) = 4y^3\cos(z)$. Hence we have

$$f(x,y,z) = x^2z^3 + xe^y + y^4\cos(z) + h(z)$$

for some function h(z). We also know that we should have $f_z(x,y,z) = 3x^2z^2 - y^4\sin(z)$. This means $h'(z) = 0$. Hence we have

$$f(x,y,z) = x^2z^3 + xe^y + y^4\cos(z) + C$$

for some constant C. Hence the work done by the force $\mathbf{F}(x,y,z)$ over the curve parameterized by $\mathbf{r}(t)$ from $t = 0$ to $t = \frac{1}{2}$ can be calculated as follows:

$$\int_{\mathbf{r}(0)}^{\mathbf{r}(\frac{1}{2})} \mathbf{F}\, d\mathbf{r} = \int_{(0,0,-1)}^{(1,\frac{1}{8},0)} \mathbf{F}\, d\mathbf{r} = f(1,\frac{1}{8},0) - f(0,0,-1) = \sqrt[8]{e} + \frac{1}{4096}$$

3. Among all smooth simple closed curves in the plane oriented counterclockwise, find the one along which the work done by

$$\mathbf{F}(x, y) = (\frac{4}{3}y^3 - 20y + 5)\mathbf{i} + (1 + 5x - 3x^3)\mathbf{j}$$

is greatest and calculate the area of the region enclosed by this smooth simple closed curve.

Solution : Let C be a smooth simple closed curve in the plane oriented counterclockwise and R be the region enclosed by the curve C. Let $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$. Then by Green's Theorem we have

$$(\text{Work done over } C) = \oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \iint_R \text{curl}(\mathbf{F})\cdot\mathbf{k}\, dA = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R \big((5 - 9x^2) - (4y^2 - 20)\big)\, dA = \iint_R (25 - 9x^2 - 4y^2)\, dA = (*)$$

Define a region $R_0 = \big\{(x, y) \,\big|\, 9x^2 + 4y^2 \le 25\big\}$ and let C_0 be the boundary of the region R_0 oriented counterclockwise. Then we have

$$(*) = \iint_{R\cap R_0} (25 - 9x^2 - 4y^2)\, dA + \iint_{R-R_0} (25 - 9x^2 - 4y^2)\, dA$$

$$\le \iint_{R\cap R_0} (25 - 9x^2 - 4y^2)\, dA \le \iint_{R_0} (25 - 9x^2 - 4y^2)\, dA =$$

$$= \oint_{C_0} \mathbf{F}\cdot\mathbf{T}\, ds = (\text{Work done over } C_0)$$

Hence among all smooth curves the greatest work is done over C_0. Hence the area enclosed by C_0 is the area of R_0 which could be calculated by making the subs $u = 3x$ and $v = 2y$ as follows

$$\iint_{R_0} dA = \iint_{\{(u,v)\,|\,u^2+v^2\le 25\}} J(u, v)\, du\, dv = \iint_{\{(u,v)\,|\,u^2+v^2\le 25\}} \frac{1}{6}\, du\, dv = \frac{25\pi}{6}$$

4. Let C be a smooth curve that encloses a region R such that the area of the region R is 7π and the interior of the region R contains the rectangle

$$D = \{(x,y) \,|\, -1 \le x \le 1 \text{ and } -1 \le y \le 1\}.$$

Compute the outward flux of the vector field

$$\mathbf{F}(x,y) = \left(\frac{2x+y}{x^2+y^2} + 3x + 6y \right) \mathbf{i} + \left(\frac{2y-x}{x^2+y^2} + 5x + 7y \right) \mathbf{j}$$

across the curve C.

Solution : Define $D_0 = \big\{(x,y) \,|\, x^2+y^2 \le 1\big\}$ and let C_0 be the boundary of D_0 oriented counterclockwise. Then D_0 is a region included in the region D hence it is also included in the region R. First note that the vector field $\mathbf{F}(x,y)$ is defined on the region $R - D_0$ and second note the following equality

$$\operatorname{div}(\mathbf{F}) = \left(\frac{2x^2+2y^2-4x^2-2xy}{\left(x^2+y^2\right)^2} + 3 \right) + \left(\frac{2x^2+2y^2-4y^2+2xy}{\left(x^2+y^2\right)^2} + 7 \right) = 10$$

Hence

$$\iint\limits_{R-D_0} \operatorname{div}(\mathbf{F}) \, dA = 10 \iint\limits_{R-D_0} dA = 10 \, (\text{area of } R - D_0) =$$

$$= 10 \left((\text{area of } R) - (\text{area of } D_0) \right) = 10 \, (7\pi - \pi) = 60\pi$$

We also have

$$\oint\limits_{C_0} \mathbf{F} \cdot \mathbf{n} \, ds = \oint\limits_{C_0} - \left(\frac{2y-x}{x^2+y^2} + 5x + 7y \right) dx + \left(\frac{2x+y}{x^2+y^2} + 3x + 6y \right) dy =$$

$$= \oint\limits_{C_0} -(2y - x + 5x + 7y) \, dx + (2x + y + 3x + 6y) \, dy = \oint\limits_{C_0} -(4x+9y) \, dx + (5x+7y) \, dy =$$

$$= \iint\limits_{D_0} (5 - (-9)) \, dA = 14 \iint\limits_{D_0} dA = 14 \, (\text{area of } D_0) = 14\pi$$

Now we have

$$\oint\limits_{C} \mathbf{F} \cdot \mathbf{n} \, ds - \oint\limits_{C_0} \mathbf{F} \cdot \mathbf{n} \, ds = \iint\limits_{R-D_0} \operatorname{div}(\mathbf{F}) \, dA$$

thus the outward flux of the vector field $\mathbf{F}(x,y)$ across the curve C is 74π.

5. Find the area of the surface $z = 2xy$ inside the cylinder $x^2 + y^2 = 9$.

Solution : Define $f(x, y, z) = z - 2xy$ and let S be the surface $f(x, y, z) = 0$ inside the cylinder $x^2 + y^2 = 9$. Choose $\mathbf{p} = \mathbf{k}$ and let R be shadow region of the surface S in the xy-plane with unit normal vector $\mathbf{p}$. Then

$$(\text{Area of } S) = \iint\limits_S d\sigma = \iint\limits_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint\limits_R \frac{|-2y\mathbf{i} - 2x\mathbf{j} + \mathbf{k}|}{|(-2y\mathbf{i} - 2x\mathbf{j} + \mathbf{k}) \cdot \mathbf{k}|}\, dA =$$

$$= \iint\limits_R \sqrt{4y^2 + 4x^2 + 1}\, dA = \int_0^{2\pi} \int_0^3 \sqrt{4r^2 + 1}\, r\, dr\, d\theta =$$

$$= \int_0^{2\pi} \left(\frac{1}{12}(4r^2 + 1)^{\frac{3}{2}} \Big|_0^3 \right) d\theta = \int_0^{2\pi} \frac{1}{12} \left(37^{\frac{3}{2}} - 1 \right) d\theta = \frac{\pi}{6} \left(37^{\frac{3}{2}} - 1 \right)$$