Solution to MATH 116 MIDTERM 1 Examination, 21/06/2008

1. a) Use the $\varepsilon - \delta$ definition of limit to show that

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{x^2+y^4} = 0\,.$$

Solution. Our aim is to show that
for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < \sqrt{x^2+y^2} < \delta \qquad \Longrightarrow \qquad \left|\frac{x^4y^4}{x^2+y^4} - 0\right| < \varepsilon\,. \qquad (*)$$

We have,

$$\left|\frac{x^4y^4}{x^2+y^4} - 0\right| = x^4\frac{y^4}{x^2+y^4} \le x^4 \le (x^2+y^2)^2 < \delta^4\,.$$

Taking $\delta = \varepsilon^{1/4}$ provides the implication $(*)$.

Note that the choice $\delta = \varepsilon^{1/4}$ is not unique. For example, any positive δ that is less than $\varepsilon^{1/4}$, or $\delta \le \min\{1, \varepsilon\}$ will provide $(*)$ as well.

1. b) Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3}$$

does not exist.

Solution. Since

$$\lim_{(x,y)\to(0,0),y=x} \frac{x^4y^4}{(x^2+y^4)^3} = \lim_{x\to 0} \frac{x^8}{(x^2+x^4)^3} = 0$$

and

$$\lim_{(x,y)\to(0,0),x=y^2} \frac{x^4y^4}{(x^2+y^4)^3} = \lim_{y\to 0} \frac{y^{12}}{(y^4+y^4)^3} = \frac{1}{8} \ne 0,$$

then, by the Two Path Test, the limit does not exist.

Solution to MATH 116 MIDTERM 1 Examination, 21/06/2008

2. a) Let $f(t)$ be a differentiable function. If $u(x,y) = f\left(\frac{x}{y}\right)$ for $y \neq 0$, prove that $u(x,y)$ satisfies the partial-differential equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0\,.$$

Solution. If $u(x,y) = f(t)$ and $t = \frac{x}{y}$ then, by the Chain Rule,

$$\frac{\partial u}{\partial x} = f'(t)\frac{\partial t}{\partial x} = \frac{1}{y}f'\left(\frac{x}{y}\right), \qquad \frac{\partial u}{\partial y} = f'(t)\frac{\partial t}{\partial y} = -\frac{x}{y^2}f'\left(\frac{x}{y}\right)\,.$$

Hence,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{1}{y}f'\left(\frac{x}{y}\right) - y\frac{x}{y^2}f'\left(\frac{x}{y}\right) = 0\,.$$

2. b) Find function $f(t)$ such that $f\left(\frac{x}{y}\right) = u(x,y)$, where $u_x\left(x, \frac{1}{x}\right) = \frac{1}{x}$ and $u(1,1) = 2$.

Solution. If $u(x,y) = f(t)$ and $t = \frac{x}{y}$ then, by the Chain Rule,

$$\frac{\partial u}{\partial x} = f'(t)\frac{\partial t}{\partial x} = \frac{1}{y}f'\left(\frac{x}{y}\right)\,.$$

Therefore,

$$\frac{\partial u}{\partial x}\bigg|_{(x,\frac{1}{x})} = u_x\left(x, \frac{1}{x}\right) = xf'(x^2)\,.$$

On the other hand, it is given that

$$\frac{\partial u}{\partial x}\bigg|_{(x,\frac{1}{x})} = u_x\left(x, \frac{1}{x}\right) = \frac{1}{x}\,.$$

Thus, $f'(x^2) = \frac{1}{x^2}$, or the same, $f(t)$ is a differentiable function such that $f'(t) = \frac{1}{t}$ for any $t > 0$. It follows that $f(t)$ can be taken as $f(t) = \ln|t| + C$, $t \neq 0$, where C is some constant. To find C we use

$$2 = u(1,1) = f(1) = \ln|1| + C = C.$$

Therefore, $f(t) = \ln|t| + 2$.

Solution to MATH 116 MIDTERM 1 Examination, 21/06/2008

3.) Let $w(x, y) = f(x - cy)$, where c is some constant and $f'(0) = 1$.

a) Calculate the directional derivative of w at the point $(c, 1)$ in the direction of $c\vec{i} + \vec{j}$.

Solution. If $w = f(t)$ and $t = x - cy$ then, by the Chain Rule,

$$\frac{\partial w}{\partial x} = f'(t)\frac{\partial t}{\partial x} = f'(t), \qquad \frac{\partial w}{\partial x}\Big|_{(c,1)} = f'(0) = 1,$$

and

$$\frac{\partial w}{\partial y} = f'(t)\frac{\partial t}{\partial y} = -cf'(t), \qquad \frac{\partial w}{\partial y}\Big|_{(c,1)} = cf'(0) = -c.$$

Thus, the gradient of function $w(x, y)$ at $(c, 1)$ is

$$\bigtriangledown w(c, 1) = \vec{i} - c\vec{j}$$

and the directional derivative of w at the point $(c, 1)$ in the direction of $c\vec{i} + \vec{j}$ is

$$D_{\frac{c\vec{i}+\vec{j}}{\sqrt{c^2+1}}} w\Big|_{(c,1)} = (\vec{i} - c\vec{j}) \circ \left(\frac{c}{\sqrt{c^2+1}}\vec{i} + \frac{1}{\sqrt{c^2+1}}\vec{j}\right) = 0.$$

b) Find constant(s) c such that the maximum directional derivative of w at $(c, 1)$ (that is, the derivative in the direction where w increases most rapidly at $(c, 1)$) is 7.

Solution. The directional derivative of w is maximum in the direction of $\bigtriangledown w(c, 1)$. Then, the maximum directional derivative of w at $(c, 1)$ is $|\bigtriangledown w(c, 1)| = \sqrt{1 + c^2}$. It is equal to 7 for $c = \sqrt{48}$ and $c = -\sqrt{48}$.

Solution to MATH 116 MIDTERM 1 Examination, 21/06/2008

4. a) A surface S in the xyz-space is given by the equation

$$x^3 + xz^2 + yz^3 + y^2 = 68\,.$$

Find an equation for the tangent plane to S at the point $(1, 2, 3)$.

Solution. The tangent plane is the plane through the point $(1, 2, 3)$ perpendicular to the gradient of function $f(x, y, z) = x^3 + xz^2 + yz^3 + y^2 - 68$ at the point $(1, 2, 3)$. The gradient is

$$\bigtriangledown f(1, 2, 3) = ((3x^2 + z^2)\vec{i} + (z^3 + 2y)\vec{j} + (2xz + 3yz^2)\vec{k})_{(1,2,3)} = 12\vec{i} + 31\vec{j} + 60\vec{k}.$$

The tangent plane is therefore

$$12(x - 1) + 31(y - 2) + 60(z - 3) = 0.$$

4. b) Find the linearization $L(x, y)$ of the function $f(x, y) = \sin x \cos y$ at the point $P\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$. Then find an upper bound for the magnitude of the error E in the approximation $f(x, y) \approx L(x, y)$ over the rectangle

$$R: \qquad \left|x - \frac{\pi}{4}\right| \leq 0.2, \qquad \left|y - \frac{\pi}{4}\right| \leq 0.1\,.$$

Solution. Since

$$f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \frac{1}{2}, \quad f_x\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \cos x \cos y\,\Big|_{(\frac{\pi}{4}, \frac{\pi}{4})} = \frac{1}{2}, \quad f_y\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = -\sin x \sin y\,\Big|_{(\frac{\pi}{4}, \frac{\pi}{4})} = -\frac{1}{2},$$

then the linearization of $f(x, y)$ at the point $P\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ is

$$L(x, y) = \frac{1}{2} + \frac{1}{2}\left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(y - \frac{\pi}{4}\right) = \frac{1}{2} + \frac{x}{2} - \frac{y}{2}\,.$$

We use the inequality

$$|E(x, y)| \leq \frac{1}{2}M(|x - \frac{\pi}{4}| + |y - \frac{\pi}{4}|)^2$$

to estimate an upper bound for the error $E(x, y)$ in the approximation $f(x, y) \approx L(x, y)$. Since

$$f_{xx}(x, y) = -\sin x \cos y, \quad f_{xy}(x, y) = -\cos x \sin y, \quad f_{yy}(x, y) = -\sin x \cos y,$$

then $|f_{xx}(x, y)| \leq 1$, $|f_{xy}(x, y)| \leq 1$ and $|f_{yy}(x, y)| \leq 1$ for any $(x, y) \in R$, that implies that M can be taken as 1. Therefore, for any $(x, y) \in R$,

$$|E(x, y)| \leq \frac{1}{2}M(|x - \frac{\pi}{4}| + |y - \frac{\pi}{4}|)^2 \leq \frac{1}{2}(0.2 + 0.1)^2 = 0.045.$$

Solution to MATH 116 MIDTERM 1 Examination, 21/06/2008

5.) Let $f(x,y) = x^3 + y^3 + 3x^2 - 18y^2 + 81y + 5$. Find the critical points of $f(x,y)$, and classify each point as a local maximum, a local minimum, or a saddle point.

Solution. Since $f(x,y)$ is differentiable in the whole xy-plane then the critical points of function $f(x,y)$ are points where f_x and f_y are simultaneously zero. This leads to

$$f_x(x,y) = 3x^2 + 6x = 3x(x+2) = 0,$$

$$f_y(x,y) = 3y^2 - 36y + 81 = 3(y-3)(y-9) = 0.$$

Therefore, the critical points are $(0,3)$, $(0,9)$, $(-2,3)$, $(-2,9)$.

We have,

$$f_{xx}(x,y) = 6x + 6 = 6(x+1), \quad f_{yy}(x,y) = 6y - 36 = 6(y-6), \quad f_{xy}(x,y) = 0,$$

and the discriminant

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6(x+1) & 0 \\ 0 & 6(y-6) \end{vmatrix} = 36(x+1)(y-6)\,.$$

<u>Point $(0,3)$</u>: Since $D(0,3) = -108 < 0$ then $(0,3)$ is a saddle point of function f.

<u>Point $(0,9)$</u>: Since $D(0,9) = 108 > 0$ and $f_{xx}(0,9) = 6 > 0$ then function f has a local minimum value at the point $(0,9)$.

<u>Point $(-2,3)$</u>: Since $D(-2,3) = 108 > 0$ and $f_{xx}(-2,3) = -6 < 0$ then function f has a local maximum value at the point $(-2,3)$.

<u>Point $(-2,9)$</u>: Since $D(-2,9) = -108 < 0$ then $(-2,9)$ is a saddle point of function f.