Due date: June 19, 2009, Friday

Math 116 Calculus – Homework # 1 – Solutions

Remark: Solve all the problems. One of the problems, randomly chosen, will be graded.

Q-1) Can the function $f(x,y) = \dfrac{\sin x \, \sin^3 y}{1 - \cos(x^2+y^2)}$ be defined at the origin in such a way that it becomes continuous there? If so, how?

Solution: Limit of the expression as (x,y) goes to $(0,0)$ along x-axis (or y-axis) is zero but limit along $y = x$ direction is $1/2$. So this function cannot be continuously extended to the origin.

Q-2) Directional derivative of a function $f(x,y)$ at the point $(-1,1)$ in the direction $\vec{i}+\vec{j}$ is 5, and in the direction $\vec{i}-\vec{j}$ is -5. Find the directional derivative of $f(x,y)$ at $(-1,1)$ in the direction of $-\vec{i}-2\vec{j}$.

Solution: $\vec{u} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $\vec{v} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $\vec{w} = (\frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}})$ are the unit vectors in the directions of $\vec{i}+\vec{j}$, $\vec{i}-\vec{j}$ and $-\vec{i}-2\vec{j}$ respectively.

We are given that $\nabla f(-1,1)\cdot\vec{u} = 5$ and $\nabla f(-1,1)\cdot\vec{v} = -5$. This amounts to saying:
$f_x(-1,1)\,\frac{1}{\sqrt{2}} + f_y(-1,1)\,\frac{1}{\sqrt{2}} = 5$ and
$f_x(-1,1)\,\frac{1}{\sqrt{2}} - f_y(-1,1)\,\frac{1}{\sqrt{2}} = -5.$

From these equations we find that $f_x(-1,1) = 0$ and $f_y(-1,1) = 5\sqrt{2}$.

We are asked to calculate $\nabla f(-1,1)\cdot\vec{w} = f_x(-1,1)\,\frac{-1}{\sqrt{5}} + f_y(-1,1)\,\frac{-2}{\sqrt{5}}$. Putting in the values of $f_x(-1,1)$ and $f_y(-1,1)$ into this, we find that the required directional derivative is $-2\sqrt{2}\sqrt{5}$.

Q-3) The plane $y = -1$ intersects the elliptic paraboloid $2x^2 + y^2 - z = 0$ in a parabola. Write parametric equations for the tangent line to this parabola at the point $(2, -1, 9)$.

Solution: Let $f(x, y, z) = 2x^2+y^2-z$ and $g(x, y, z) = y+1$. The parabola in the question is the curve of intersection of the two level surfaces $f = 0$ and $g = 0$. The tangent line of this parabola, at any point (x, y, z) on it, points in the direction of $\vec{n}(x, y, z)$ where

$$\vec{n}(x,y,z) = \nabla f \times \nabla g = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4x & 2y & -1 \\ 0 & 1 & 0 \end{vmatrix} = (1, 0, 4x).$$

Parametric equations of the tangent line to this parabola at the point $(2, -1, 9)$ is given by $(x(t), y(t), z(t)) = (2, -1, 9) + t\,\vec{n}(2, -1, 9)$, $t \in \mathbb{R}$, or equivalently as
$x(t) = 2 + t$,
$y(t) = -1$,
$z(t) = 9 + 8t$, where $t \in \mathbb{R}$.

Q-4) Let

$$f(x,y) = \begin{cases} \frac{2xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Calculate and compare $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.

Solution:

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0} \frac{0}{x} = 0.$$

Similarly $f_y(0, 0) = 0$. This gives

$$f_x(x,y) = \begin{cases} \dfrac{2y\,(x^4 - y^4 + 4\,x^2y^2)}{(x^2+y^2)^2}, & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Using this we can calculate $f_{xy}(0, 0)$ as follows:

$$\begin{aligned} f_{xy}(0,0) &= \lim_{y\to 0} \frac{f_x(0,y) - f_x(0,0)}{x} \\ &= \lim_{y\to 0} \frac{\frac{-2y^5}{y^4} - 0}{y} \\ &= -2. \end{aligned}$$

Similarly we have

$$f_y(x,y) = \begin{cases} \dfrac{2x\,(x^4 - y^4 - 4\,x^2y^2)}{(x^2+y^2)^2}, & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0) \end{cases}$$

and using this as above we find $f_{yx}(0, 0) = 2$.

Q-5) Find $\dfrac{\partial^3}{\partial t^2 \partial s} f(s^2 - \cos^2 3t, e^{s^2} + 5t^2)$ in terms of the partial derivatives of f.

Solution: Note that for any function $g = g(s^2 - \cos^2 3t, e^{s^2} + 5t^2)$ we have

$$\frac{\partial}{\partial t} g = (g_x, g_y) \cdot (3 \sin 6t, 10t) .$$

We first find that

$$\frac{\partial}{\partial s} f = 2s \left(f_x + c \, f_y \right)$$

where $c = e^{s^2}$. Then

$$\begin{aligned}
\frac{\partial^2}{\partial t \partial s} f &= \frac{\partial}{\partial t} \frac{\partial f}{\partial s} \\
&= 2s \left[(f_{xx}, f_{xy}) \cdot (3 \sin 6t, 10t) + c \left(f_{yx}, f_{yy} \right) \cdot (3 \sin 6t, 10t) \right] \\
&= (6s) \sin 6t \, f_{xx} + (20s) \; t \, f_{xy} + (6sc) \sin 6t \, f_{yx} + (20cs) \; t \, f_{yy} \\
\frac{\partial^3}{\partial t^2 \partial s} f &= \left\{ (36s) \cos 6t \, f_{xx} + (6s) \sin 6t \; (f_{xxx}, f_{xxy}) \cdot (3 \sin 6t, 10t) \right\} \\
&\quad + \left\{ (20s) \; f_{xy} + (20s) \; t \; (f_{xyx}, f_{xyy}) \cdot (3 \sin 6t, 10t) \right\} \\
&\quad + \left\{ (36sc) \cos 6t \, f_{yx} + (6sc) \sin 6t \; (f_{yxx}, f_{yxy}) \cdot (3 \sin 6t, 10t) \right\} \\
&\quad + \left\{ (20cs) \; f_{yy} + (20cs) \; t \; (f_{yyx}, f_{yyy}) \cdot (3 \sin 6t, 10t) \right\} .
\end{aligned}$$
