Date: July 4, 2009, Saturday
Time: 10:00-12:00

NAME:..

STUDENT NO:..

SECTION NUMBER: ....

Math 116 Intermediate Calculus III – Midterm Exam II – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit. Without the correct **section number**, your grade may not be entered in SAPS.

Q-1) Evaluate $\displaystyle\int\int_R \arctan\left(\frac{y}{x}\right)\, dA$, where

$$R = \{(x, y) \in \mathbb{R}^2 \mid 1 \le x^2 + y^2 \le 4,\ 0 \le y \le x\}.$$

Solution (Müfit Sezer):

$$\begin{aligned}\int\int_R \arctan\left(\frac{y}{x}\right)\, dA &= \int_0^{\pi/4}\int_1^2 \theta\, r\, dr\, d\theta \\ &= \left(\frac{1}{2}\theta^2\Big|_0^{\pi/4}\right)\left(\frac{1}{2}r^2\Big|_1^2\right) \\ &= \frac{3\pi^2}{64}.\end{aligned}$$

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Q-2) Integrate the function $f(x,y,z) = y$ over the region between the planes $z + x + 2y = 4$ and $2z + 2x + y = 8$ in the first octant.

Solution (Metin Gürses):

$$
\begin{aligned}
\text{Volume} &= \int_0^4 \int_0^{4-z} \int_{2-x/2-z/2}^{8-2x-2z} y \, dy \, dx \, dz \\
&= \int_0^4 \int_0^{4-z} \left(\frac{y^2}{2} \Big|_{2-x/2-z/2}^{8-2x-2z} \right) dx \, dz \\
&= \int_0^4 \int_0^{4-z} \left(30 - 15\,z - 15\,x + \frac{15}{8}\,z^2 + \frac{15}{4}\,zx + \frac{15}{8}\,x^2 \right) dx \, dz \\
&= \int_0^4 \left(30\,x - 15\,zx - \frac{15}{2}\,x^2 + \frac{15}{8}\,z^2 x + \frac{15}{8}\,zx^2 + \frac{5}{8}\,x^3 \Big|_0^{4-z} \right) dz \\
&= \int_0^4 \left(40 - 30\,z + \frac{15}{2}\,z^2 - \frac{5}{8}\,z^3 \right) dz \\
&= \left(40\,z - 15\,z^2 + \frac{5}{2}\,z^3 - \frac{5}{32}\,z^4 \Big|_0^4 \right) \\
&= 40.
\end{aligned}
$$

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Q-3) Evaluate

$$\int_0^2 \int_{\sqrt{2y-y^2}}^{\sqrt{4y-y^2}} \int_y^4 dz\,dx\,dy + \int_2^4 \int_0^{\sqrt{4y-y^2}} \int_y^4 dz\,dx\,dy.$$

Solution (Sinan Sertöz):

$$\begin{aligned}
\int_0^2 \int_{\sqrt{2y-y^2}}^{\sqrt{4y-y^2}} \int_y^4 dz\,dx\,dy \quad + & \quad \int_2^4 \int_0^{\sqrt{4y-y^2}} \int_y^4 dz\,dx\,dy \\
= & \int_0^{\pi/2} \int_{2\sin\theta}^{4\sin\theta} \int_{r\sin\theta}^4 r\,dz\,dr\,d\theta \\
= & \int_0^{\pi/2} \int_{2\sin\theta}^{4\sin\theta} (4r - r^2\sin\theta) dr\,d\theta \\
= & \int_0^{\pi/2} \left(2r^2 - \frac{1}{3} r^3 \sin\theta \Big|_{2\sin\theta}^{4\sin\theta} \right) d\theta \\
= & \int_0^{\pi/2} \left(24\sin^2\theta - \frac{56}{3} \sin^4\theta \right) d\theta \\
= & \int_0^{\pi/2} \left(5 - \frac{8}{3} \cos 2\theta - \frac{7}{3} \cos 4\theta \right) d\theta \\
= & \left(5\theta - \frac{8}{6} \sin 2\theta - \frac{7}{12} \sin 4\theta \Big|_0^{\pi/2} \right) \\
= & \frac{5\pi}{2}.
\end{aligned}$$

Also you may observe that this integral is calculating the volume of a very symmetrical space figure. By elementary methods you can easily find its volume to be $\frac{5\pi}{2}$.

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Q-4) Integrate $f(x,y,z) = \dfrac{1}{(x^2+y^2+z^2)^2}$ on the region in the first octant between $x^2+y^2+z^2=1$ and $x+y+z=9$.

Solution (Özgün Ünlü):

$$\begin{aligned}
&\int\int\int \frac{dxdydz}{(x^2+y^2+z^2)^2} \\
= &\int_0^{\pi/2}\int_0^{\pi/2}\int_1^{9/(\sin\phi\cos\theta+\sin\phi\sin\theta+\cos\phi)} \frac{\rho^2\sin\phi}{\rho^4}\,d\rho\,d\phi\,d\theta \\
= &\int_0^{\pi/2}\int_0^{\pi/2}\int_1^{9/(\sin\phi\cos\theta+\sin\phi\sin\theta+\cos\phi)} \frac{\sin\phi}{\rho^2}\,d\rho\,d\phi\,d\theta \\
= &\int_0^{\pi/2}\int_0^{\pi/2} \sin\phi\left(-\frac{1}{\rho}\Big|_1^{9/(\sin\phi\cos\theta+\sin\phi\sin\theta+\cos\phi)}\right)d\theta\,d\phi \quad \text{(Note change of order.)} \\
= &\int_0^{\pi/2}\int_0^{\pi/2}\left(\sin\phi-\frac{1}{9}\sin^2\phi\cos\theta-\frac{1}{9}\sin^2\phi\sin\theta-\frac{1}{9}\sin\phi\cos\phi\right)d\theta\,d\phi \\
= &\int_0^{\pi/2}\int_0^{\pi/2}\left((\sin\phi-\frac{1}{9}\sin\phi\cos\phi)-\frac{1}{9}\sin^2\phi(\cos\theta+\sin\theta)\right)d\theta\,d\phi \\
= &\int_0^{\pi/2}\left((\sin\phi-\frac{1}{9}\sin\phi\cos\phi)\theta-\frac{1}{9}\sin^2\phi(\sin\theta-\cos\theta)\Big|_0^{\pi/2}\right)d\phi \\
= &\int_0^{\pi/2}\left(\frac{\pi}{2}\sin\phi-\frac{\pi}{18}\sin\phi\cos\phi-\frac{2}{9}\sin^2\phi\right)d\phi \\
= &\left(-\frac{\pi}{2}\cos\phi-\frac{\pi}{36}\sin^2\phi-\frac{\phi}{9}+\frac{1}{18}\sin 2\phi\Big|_0^{\pi/2}\right) \\
= &\frac{5\pi}{12}.
\end{aligned}$$

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Q-5) Let R be a region in xy-plane, and let T be a transformation mapping xy-plane onto uv-plane given as

$$\begin{aligned} u &= 3x - 7y \\ v &= 7x - 3y. \end{aligned}$$

Find a function $f(x, y)$ so that

$$\int\int_R f\, dA = \text{Area}(T(R)).$$

Solution (Hamza Yeşilyurt):

$$\begin{aligned} \text{Area}(T(R)) &= \int\int_{T(R)} dudv \\ &= \int\int_R \frac{\partial(u, v)}{\partial(x, y)}\, dx\, dy. \end{aligned}$$

Hence we can choose $f(x, y) = \dfrac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 3 & -7 \\ 7 & -3 \end{vmatrix} = 40.$