Date: January 17, 2009, Saturday NAME:...
Time: 10:00-12:00
Ali Sinan Sertöz STUDENT NO:...

Math 123 Abstract Mathematics I – Final Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. A correct answer without proper reasoning may not get any credit. For this exam take $\mathbb{N} = \{1, 2, \ldots\}$.

Q-1) Write in plain words the negation of the following two statements:
(a) For every positive integer d, every finite group G whose order is divisible by d has a subgroup of order d.
(b) There exists an $\epsilon > 0$ such that for every $\delta > 0$ we can find two points $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$.

Solution:
(a) There exists a positive integer d and a finite group G whose order is divisible by d such that G has no subgroup of order d.
(b) For every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $x, y \in \mathbb{R}$ either $|x - y| \geq \delta$ or $|f(x) - f(y)| < \epsilon$. (This last part is equivalent to $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.)

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Q-2) Prove or disprove that the set of all finite subsets of $\mathbb{N}$ is uncountable.

Solution: Let H be the set of all finite subsets of $\mathbb{N}$. We will show that H is countable. For this we will find an injection from H into $\mathbb{N}$.

Let the prime numbers be ordered as $p_1, p_2, \ldots$.

Define $\phi : H \to \mathbb{N}$ as follows: If $A = \{a_1, \ldots, a_n\}$ is a finite subset of $\mathbb{N}$, then define $\phi(A) = p_{a_1} \cdots p_{a_n}$.

It is clear that ϕ is an injection.

H being isomorphic (set theoretically) to the subset $\phi(H)$ of $\mathbb{N}$ is itself countable.

Other solutions from your exam papers:

• $\phi(A) = p_1^{a_1} \cdots p_n^{a_n}$ is a nice variation of the above proof.

• We can write every finite set as $A = \{a_1, \ldots, a_n\}$ where $a_1 < \cdots < a_n$. Send A to the rational number $0.a_1 a_2 \cdots a_n$, where we just juxtapose the integers to form the mentioned rational number. Now observe that only countably many finite sets will map onto the same rational number. Since H maps into rationals, it is then the union of countably many countable sets.

• Order all finite sets with lex ordering. Then H is the union of countably many countable sets.

• Let H_k be the collection of all finite sets whose maximal number is k. Clearly H_k is finite and $H = \bigcup_{k=1}^{\infty} H_k$ is countable. Brilliant!

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Q-3) Which, if any, of the following numbers belong to the Cantor set?

$$\frac{35}{108}, \frac{70}{108}, \frac{105}{108}.$$

Solution: $35 = (1022)_3$, $108 = (11000)_3$. Using long division we find that

$$\frac{35}{108} = 0._3 0222020202\ldots$$

so is an element of the Cantor set. On the other hand

$$\frac{70}{108} = 2 \times \frac{35}{108} = (2)_3 \times 0._3 0222020202\cdots = 0._3 1221111\ldots$$

so is not an element of the Cantor set, but

$$\frac{105}{108} = 3 \times \frac{35}{108} = (10)_3 \times 0._3 0222020202\cdots = 0._3 222020202\ldots$$

and is an element of the Cantor set.

The best way to see that $\frac{70}{108}$ is not in the Cantor set is to observe that $\frac{1}{3} < \frac{70}{108} < \frac{2}{3}$. (This is also from your exam solutions.)

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Q-4) Let $A \subset [0,1]$ be an infinite set. Prove or disprove the following statement:

$$\forall\, x_0 \in [0,1],\ \exists\, \epsilon > 0 \text{ such that } \forall\, x \in A \text{ either } x = x_0 \text{ or } |x - x_0| \geq \epsilon.$$

Solution: This question is not exactly what I intended to ask. As it is written the question asks if the following statement is correct:

$\forall$ infinite set $A \subset [0,1]$, $\forall\, x_0 \in [0,1]$, $\exists\, \epsilon > 0$ such that $\forall\, x \in A$ either $x = x_0$ or $|x - x_0| \geq \epsilon$.

To show that it is wrong, it suffices to find an A which is a counterexample. That is, we should prove that

$\exists$ an infinite set $A \subset [0,1]$, $\exists\, x_0 \in [0,1]$, $\forall\, \epsilon > 0$, $\exists\, x \in A$ such that $x \neq x_0$ and $|x - x_0| < \epsilon$.

For this take $A = [0,1]$, $x_0 = 0$, and for any $\epsilon > 0$ take $x = \epsilon/2$. Now clearly $x \neq x_0$ and $|x - x_0| < \epsilon$.

However, what I had in mind, but not on paper, is the following statement:

$\exists$ an infinite set $A \subset [0,1]$, $\forall\, x_0 \in [0,1]$, $\exists\, \epsilon > 0$ such that $\forall\, x \in A$ either $x = x_0$ or $|x - x_0| \geq \epsilon$.

This statement is false. The following is a proof of its converse:

$\forall$ infinite set $A \subset [0,1]$, $\exists\, x_0 \in [0,1]$ such that $\forall\, \epsilon > 0$, $\exists\, x \in A$ such that $0 < |x - x_0| < \epsilon$.

For this we apply the following procedure. Let $I_0 = [0,1]$.

Assuming that $I_n = [a_n, b_n]$ is defined, we define $I_{n+1} = [a_{n+1}, b_{n+1}] = [a_n, \dfrac{a_n + b_n}{2}]$ if $[a_n, \frac{a_n+b_n}{2}]$ contains infinitely many elements of A, and define $I_{n+1} = [a_{n+1}, b_{n+1}] = [\dfrac{a_n + b_n}{2}, b_n]$ otherwise.

We observe that $a_0 \leq a_1 \leq a_2 \leq \cdots$, $b_0 \geq b_1 \geq b_2 \geq \cdots$ and $a_n < b_n$ for all n.

The sequence $\{a_n\}$ is a bounded increasing sequence so has a limit a. Similarly the sequence $\{b_n\}$ is a bounded decreasing sequence and has a limit b.

Clearly $a_n \leq a \leq b \leq b_n$ for all n.

If we use $\ell(I_n)$ to denote the length of I_n, we see that $\ell(I_n) = b_n - a_n = 1/2^n$.

Let $\epsilon > 0$ be given.

Choose n such that $0 < 1/2^n < \epsilon$. Then $b - a \leq b_n - a_n = \ell(I_n) = 1/2^n < \epsilon$. This forces $a = b$.

Let $x_0 = a$. Since $I_0 \supset I_1 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$, $x_0 \in I_n$ for all n.

Now observe that for any $x \in I_n$, $|x - x_0| \leq b_n - a_n < \epsilon$. Since $A \cap I_n$ contains infinitely many elements, we can choose $x \in A \cap I_n$ different than x_0. This proves the statement which we claimed to be true. (The name of this statement is Bolzano-Weierstrass Theorem.)

NAME: STUDENT NO:

Q-5) Let G be a finite group and H a subgroup with the property that $i(H)$ is the smallest prime p dividing the order of G. Show that H is a normal subgroup of G.
Hint: Show that G permutes the set of right cosets of H and that the kernel must be contained in H. Now use Lagrange's theorem together with the fact that no prime larger than or equal to p can divide $(p-1)!$.

Solution: Let K be the set of right cosets of H in G. The cardinality of K is $i(H) = p$. (Here $i(H) = o(G)/o(H)$ and is called the index of H in G.) The symmetric group S_p acts on K by simply permuting its elements. Each element of G also permutes elements of K by simply multiplying each right coset from the right and hence sending it onto another right coset, not necessarily distinct than the original one. This defines a map $\phi : G \to S_p$. Check that this defines a homomorphism. We know that $\phi(G)$ is a subgroup of S_p, so $o(G)$ divides the order of S_p which is $p!$.

If $a \in \ker\phi$. Then a leaves each right coset of H fixed, in particular $H = Ha$, so $a \in H$. Hence $\ker\phi$ is a subgroup of H and its order must divide the order of H. Let $m\, o(\ker\phi) = o(H)$ for some positive integer m.

Since $o(H)|o(G)$, m must also divide the order of G. By our description of p, if q is a prime dividing m, then $q \geq p$.

We know that $\phi(G)$ is isomorphic to $G/\ker\phi$, so $o(\phi(G)) = o(G)/(o(H)/m) = m\, o(G)/o(H) = m\, i(H) = mp$. We know that this number divides $p!$, so $m|(p-1)!$.

If q is a prime dividing m, then $q|(p-1)!$ so q is a prime strictly less than p. This contradicts what we found about q above. So no prime divides m, forcing $m = 1$.

This says that $H = \ker\phi$ and hence is a normal subgroup since all kernels are normal.

Please forward any comments or questions to `sertoz@bilkent.edu.tr`
