Date: May 29, 2009, Friday NAME:..
Time: 9:30-11:30
Ali Sinan Sertöz STUDENT NO:..

Math 124 Abstract Mathematics II – Final Exam – Solutions

1	2	3	4	*Bonus*	TOTAL
25	25	25	25	*25*	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 4 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. After the exam check the course web page for solutions.

Q-1) Let $d_n(x,y) = \sum_{i=1}^{n}(x_i - y_i)^2$ for $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Which of the following statements is true? Prove your claim.
(i) d_n is a metric for every $n \geq 1$.
(ii) d_n is not a metric for any $n \geq 1$.
(iii) There exists integers n and m such that d_n is a metric but d_m is not a metric.

Solution: The second statement is true. Every d_n fails to satisfy the triangle inequality. For example take $x = (-1, 0, \ldots, 0)$, $y = (1, 0, \ldots, 0)$ and $z = (0, \ldots, 0)$. Then

$$d_n(x,y) = 4,\ d_n(x,z) = 1,\ d_n(z,y) = 1$$

and the triangle inequality fails.

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Q-2) Let $\mathcal{H}^2$ be the hyperbolic space. Let L_1 and L_2 be two lines in $\mathcal{H}^2$, intersecting at a point $P \in \mathcal{H}^2$. Describe how you would construct a line L in $\mathcal{H}^2$ ultraparallel to both L_1 and L_2.

Solution: Let V_i be the plane in $\mathbb{R}^3$, passing through the origin and such that $L_i = V_i \cap \mathcal{H}^2$, $i = 1, 2.$. Let C be the light cone and let $P_i \in C$ be one of the two points of $V_i \cap \mathcal{H}^2$, $i = 1, 2$. Now let V in $\mathbb{R}^3$ be the plane passing through P_1, P_2 and the origin. Let L in $\mathcal{H}^2$ be the line defined as $V \cap \mathcal{H}^2$. This line L is a line in $\mathcal{H}^2$ ultraparallel to both L_1 and L_2.

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Q-3) Describe an imbedding of $\mathbb{R}^2$ into $\mathbb{P}^2$. Describe the projective closures of the plane curves $y = x^2$ and $x^2 + y^2 = 1$ in $\mathbb{P}^2$ with respect to your imbedding. Show that their projective closures are isomorphic in the sense that there is an invertible polynomial map sending one to the other.

Solution: Sending (x, y) to $[x : y : 1]$ is an imbedding of $\mathbb{R}^2$ into $\mathbb{P}^2$. The projective closures of the given curves are described by the equations $zy = x^2$ and $x^2 + y^2 = z^2$. Let $[u : v : w]$ be new coordinates such that $u = x$, $v = z - y$ and $w = z + y$. Then $x^2 + y^2 - z^2 = x^2 + (z - y)(z + y) = u^2 - vw = 0$ which is the projective closure of the parabola.

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Q-4) Let X be a topological space and endow $X \times X$ with the product topology. Let $\Delta = \{(x, x) \in X \times X\}$ be the diagonal. Show that X is a Hausdorff topological space if and only if Δ is a closed set in $X \times X$.

Solution: Assume X is Hausdorff. We show that the complement of Δ in $X \times X$ is open. Let $(p, q) \in X \times X \setminus \Delta$. Since X is Hausdorff there exists disjoint open sets U and V in X such that $p \in U$ and $q \in V$. Then $U \times V$ is disjoint from Δ since any point common to both would be of the form (x, x), meaning that $x \in U$ and $x \in V$, a contradiction since U and V are disjoint. This shows that $U \times V$ is an open neighborhood of (p, q) lying totally in the complement of Δ. This proves that the complement of Δ is open, and hence Δ is closed.

Conversely assume that Δ is closed. Then its complement is open and for any two distinct points $p, q \in X$, the point (p, q) lies in the complement of Δ. Hence it has an open neighborhood $U \times V$ lying in the complement of Δ. Then U and V must be disjoint since any common point would lie on the diagonal. The open sets U and V separate the points p and q in X so X is Hausdorff.

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Bonus) If n points in the plane are not on one straight line, then there exists a straight line containing exactly two of the points.

Solution: Let d be the usual distance metric in $\mathbb{R}^2$. For any three distinct points P, Q and R, let $d(P, QR)$ denote the distance from the point P to the line QR.

Let the given points be $P_1, \ldots, P_n$. Consider the set

$$A = \{d(P_i, P_jP_k) \mid j \neq k,\ P_i \notin P_jP_k\}.$$

A is a finite set of strictly positive real numbers, so it has a minimum element. Rearranging the indices if necessary assume that $d(P_1, P_2P_3)$ is a minimum element.

We claim that P_2P_3 is the required line.

Assume not. Then there is a point say P_m lying on P_2P_3 and $m \neq 2, 3$. Let Q be the base of the perpendicular from P_1 to P_2P_3. Clearly the length of the line segment P_1Q is $d(P_1, P_2P_3)$ by definition. Of the three points P_2, P_3 and P_m, at least two of them must lie on the same side of Q on P_2P_3. Assume without loss of generality that P_1 and P_2 both lie to the right of Q. Assume P_2 lies between Q and P_3. We do not exclude the possibility that $Q = P_2$.

Let R be the foot of the perpendicular from P_2 to P_1P_3. Clearly the length of the line segment P_2R is $d(P_2, P_1P_3)$ and is strictly smaller than P_1Q which is $d(P_1, P_2P_3)$, contradicting the minimality of $d(P_1, P_2P_3)$. This proves the claim that P_2P_3 contains no other given point.