Due Date: 19 November 2015, Thursday
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:...

STUDENT NO:.......................................

Math 202 Complex Analysis – Homework 3 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are **3** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

Rules for Homework and Take-Home Exams

(1) You may discuss the problems only with your classmates or with me. In particular you may not ask your assigned questions or any related question to online forums.

(2) You may use any written source be it printed or online. Google search is perfectly acceptable.

(3) It is absolutely mandatory that you write your answers alone. Any similarity with your written words and any other solution or any other source that I happen to know is a direct violation of honesty.

(4) You must obey the usual rues of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends.

(5) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you exhibit your total understanding of the ideas involved.

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Q-1) Define a function $F(m,n) = \dfrac{1}{2\pi i}\displaystyle\int_{|z|=2} z^n(1-z)^m\,dz$, where $m,n \in \mathbb{Z}$. Find explicitly the value of $F(m,n)$.

Note that for notational convenience I have redefined $F(m,n)$ by a factor of $2\pi i$.

Solution:

We will use Cauchy Integral Formula which says that

$$f^{(k)}(z_0) = \frac{k!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz,$$

where f is analytic on and inside the closed contour C, z_0 is a point inside of C and k is a non-negative integer.

Case (1) $m,n \geq 0$.
In this case the function $z^n(1-z)^m$ is analytic inside $|z|=2$, and by the Cauchy-Goursat Theorem the integral is zero.

$$\boxed{F(m,n) = 0, \text{ if } m,n \geq 0.}$$

Case (2) $m \geq 0 > n$.
Let $f_m(z) = (1-z)^m$. Then f is analytic and

$$F(m,n) = \frac{1}{2\pi i}\int_{|z|=2} \frac{f_m(z)}{(z-0)^{(k-1)+1}},$$

where $k = -n > 0$. By CIF we get

$$F(m,n) = \frac{f_m^{(k-1)}(0)}{(k-1)!}.$$

Now we have two subcases.

Case (2.1) $0 \leq k-1 \leq m$.
In this case we have

$$f_m^{(k-1)}(0) = (-1)^{k-1}\frac{m!}{(m-k+1)!}, \quad \text{and} \quad \frac{f_m^{(k-1)}(0)}{(k-1)!} = (-1)^{k-1}\frac{m!}{(k-1)!(m-k+1)!}.$$

Thus in this case we have

$$F(m,n) = (-1)^{-n-1}\binom{m}{m+n+1}, \quad \text{if} \quad m \geq 0 > n \quad \text{and} \quad m+n \geq -1.$$

Case (2.2) $0 \leq m < k-1$.
In this case we have $f_m^{(k-1)}(z) = 0$, hence

$$F(m,n) = 0, \text{ if } m \geq 0 > n \text{ and } m+n < -1.$$

If we adopt the convention that $\binom{a}{b} = 0$ when b is not in the range 0 to a, then we can summarize the result of Case 2 as follows.

$$\boxed{F(m,n) = (-1)^{n+1}\binom{m}{m+n+1}, \text{ if } m \geq 0 > n.}$$

Case (3) $n \geq 0 > m$.

In this case we are evaluating the integral $\dfrac{(-1)^m}{2\pi i}\displaystyle\int_{|z|=2} \dfrac{z^n}{(z-1)^{(-m-1)+1}}\, dz$. Arguing as in the previous case we find that in this case we have

$$\boxed{F(m,n) = (-1)^{m}\binom{n}{m+n+1}, \text{ if } n \geq 0 > m.}$$

Case (4) $m, n < 0$.

In this case let $-m = k > 0$ and $-n = \ell$. Then we have

$$F(m,n) = \frac{1}{2\pi i}\int_{|z|=1/2} \frac{\frac{1}{(1-z)^k}}{(z-0)^{(\ell-1)+1}}\, dz + \frac{(-1)^k}{2\pi i}\int_{|z-1|=1/2} \frac{\frac{1}{z^\ell}}{(z-1)^{(k-1)+1}}\, dz.$$

Using CIF we see that the first integral is given by

$$\left.\frac{d^{\ell-1}}{dz^{\ell-1}}\right|_{z=0}\left(\frac{1}{(1-z)^k}\right) = \binom{k+\ell-2}{k-1},$$

and the second integral is equal to

$$(-1)^k \left.\frac{d^{k-1}}{dz^{k-1}}\right|_{z=1}\left(\frac{1}{z^\ell}\right) = -\binom{k+\ell-2}{k-1}.$$

Thus we see that in this final case we have

$$\boxed{F(m,n) = 0, \text{ if } m, n < 0.}$$

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Q-2) Let f be an entire function. Fix two arbitrary points $z_1 \neq z_2$ in $\mathbb{C}$. Show that the integral of f along any contour from z_1 to z_2 is the same regardless of which contour used. Thus define a function $G(z) = \int_0^z f(z)\,dz$. Show that $G'(z) = f(z)$. Let F be any antiderivative of f. Show that $\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$.

Solution:

If C_1 and C_2 are two contours from z_1 to z_2, then $C_1 - C_2$ is a closed contour and by Cauchy-Goursat Theorem, the integral of f along this loop is zero.

$$0 = \int_{C_1 - C_2} f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz.$$

This then shows that

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

To show that $G'(z) = f(z)$ we use the continuity of f at z. For any $\epsilon > 0$ let $\delta > 0$ be such that $|f(\tau) - f(z)| < \epsilon$ whenever $|\tau - z| < \delta$. Take any complex number h with $0 < |h| < \delta$, and evaluate the following integrals along a line joining z to $z + h$. Then we have

$$\begin{aligned} \left|\frac{G(z+h) - G(z)}{h} - f(z)\right| &= \left|\frac{1}{h}\int_z^{z+h} f(\tau)\,d\tau - \frac{1}{h}\int_z^{z+h} f(z)\,d\tau\right| \\ &\leq \frac{1}{|h|}\int_z^{z+h} |f(\tau) - f(z)|\,|d\tau| \\ &< \epsilon, \end{aligned}$$

which proves that $G'(z) = f(z)$.

Using the definition of G we have

$$\int_{z_1}^{z_2} f(z)\,dz = \int_0^{z_2} f(z)\,dz - \int_0^{z_1} f(z)\,dz = G(z_2) - G(z_1).$$

If F is any other antiderivative of f, then $G = F + C$ for some constant C, and we have

$$G(z_2) - G(z_1) = F(z_2) - F(z_1),$$

as claimed.

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Q-3) Evaluate the integral $\int_{|z|=1} z^n \, dz$ in two ways: (a) Use Cauchy theorems, (b) Use the definition of path integral.

Solution:

Using Cauchy Theorem, the integral is zero when $n \geq 0$, since in that case z^n is analytic around zero. If $n < 0$, then let $-n = k > 0$ and set $f(z) = 1$. The integral then becomes

$$\int_{|z|=1} \frac{f(z)}{z^{(k-1)+1}} \, dz.$$

If $k = 1$, then by Cauchy Theorem, the integral is equal to $2\pi i \, f(0) = 2\pi i$. If $k > 1$ then the integral involves the derivatives of $f = 1$, so the integral is zero. Hence we have by Cauchy Theorem

$$\int_{|z|=1} z^n \, dz = \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Using the definition of path integral requires that we start with a smooth parametrization of the contour $|z| = 1$. Let $z = e^{it}$, $t \in [0, 2\pi]$ be such a parametrization. Then $dz = ie^{it} dt$, and we get

$$\int_{|z|=1} z^n \, dz = \int_0^{2\pi} ie^{i(n+1)t} \, dt.$$

If $n = -1$, then the integrand becomes the constant i, and hence the integral is $2\pi i$. If $n \neq -1$, then using the result of the previous problem, The Fundamental Theorem of Calculus, we get

$$\int_{|z|=1} z^n \, dz = \int_0^{2\pi} ie^{i(n+1)t} \, dt = \left(\left. \frac{e^{i(n+1)t}}{n+1} \right|_0^{2\pi} \right) = 0.$$

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Q-4) Let f be an entire function whose n-the derivative is bounded in the plane. Show that f is a polynomial of degree n.

Solution:

Fix any point z_0 in the plane. We will show that $f^{(n+1)}(z_0) = 0$. This suffices to show that f is a polynomial of degree n. Let $R > 0$ be any real number, and let $M > 0$ be an upper bound for the n-th derivative of f in the plane. By Cauchy Integral Formula we have

$$f^{(n+1)}(z_0) = \frac{1}{2\pi i} \int_{|z|=R} \frac{f^{(n)}(z)}{(z - z_0)^2} \, dz.$$

Taking absolute value of both sides, we get

$$\left| f^{(n+1)}(z_0) \right| \leq \frac{M}{R}.$$

Since this holds for any $R > 0$, by sending R to infinity we see that $f^{(n+1)}(z_0) = 0$.

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Q-5) Let a, b, c be distinct complex numbers and let C be a simple closed contour containing all of them in its interior. Show that

$$\frac{1}{2\pi i}\int_C \frac{z^2}{(z-a)(z-b)(z-c)}\,dz = 1.$$

Solution: Let us define a function, for any $u, v \in \mathbb{C}$, as

$$f_{uv}(z) = \frac{z^2}{(z-u)(z-v)},\ z \in \mathbb{C}.$$

Let $r > 0$ be small such that the closed discs with radii r and centers at a, b and c totally lie inside C. Then we have

$$\frac{1}{2\pi i}\int_C \frac{z^2}{(z-a)(z-b)(z-c)}\,dz = \frac{1}{2\pi i}\int_{|z-a|=r} \frac{f_{bc}(z)}{(z-a)}\,dz + \frac{1}{2\pi i}\int_{|z-b|=r} \frac{f_{ca}(z)}{(z-b)}\,dz + \frac{1}{2\pi i}\int_{|z-c|=r} \frac{f_{ab}(z)}{(z-c)}\,dz.$$

By Cauchy Integral Formula this sum is equal to

$$f_{bc}(a) + f_{ca}(b) + f_{ab}(c).$$

Now a straightforward calculation shows that this sum is 1.