Due Date: 14 December 2015, Monday
Time: Class Time
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:.......................................

Math 202 Complex Analysis – Homework 4 – Solutions

1	2	3	4	5	TOTAL
50	50	0	0	0	100

Please do not write anything inside the above boxes!

Check that there are **2** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Submit your solutions on this booklet only. Use extra pages if necessary.**

Rules for Homework and Take-Home Exams

(1) You may discuss the problems only with your classmates or with me. In particular you may not ask your assigned questions or any related question to online forums.

(2) You may use any written source be it printed or online. Google search is perfectly acceptable.

(3) It is absolutely mandatory that you write your answers alone. Any similarity with your written words and any other solution or any other source that I happen to know is a direct violation of honesty.

(4) You must obey the usual rues of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends.

(5) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you exhibit your total understanding of the ideas involved.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

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NAME: STUDENT NO:

Q-1) Suppose the nonconstant function $f(z)$ is analytic in a domain D and continuous on its closure $\bar{D}$. If $|f(z)|$ is constant on the boundary of D, prove that $f(z)$ has a zero in D.

Solution:

If $f(z)$ has no zero in D, then by the minimum modulus theorem the minimum of $|f(z)|$ holds on the boundary. This means that $\min |f(z)| = \max |f(z)|$. Thus $|f(z)|$ is constant. This implies that $f(z)$ is constant contrary to assumption. So $f(z)$ must have a zero inside D.

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Q-2) Find the minimum and maximum values of $\left|\frac{z}{z^2+9}\right|$ on $1 \leq |z| \leq 2$.

Solution:

Since the function has no zeros inside the domain, the minimum and maximum values will hold on the boundary. We recall the triangle inequalities.

$$|z_1 + z_2| \leq |z_1| + |z_2|, \quad \text{equality holds when } z_1 = rz_2| \text{ with } r \geq 0,$$

and

$$|z_1 + z_2| \geq \left||z_1| - |z_2|\right|, \quad \text{equality holds when } z_1 = rz_2| \text{ with } r \leq 0.$$

After a short investigation we find that the maximum modulus holds at $z = \pm 2i$, and is $\frac{2}{5}$. The minimum occurs at $z = 1$ and is $\frac{1}{10}$.

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Q-3) Find the Laurent expansions of the function $f(z) = \dfrac{z-12}{z^2+z-6}$ in the following regions.

(a) $1 < |z-1| < 4$.

(b) $|z - 1 < 1$.

(c) $|z-1| > 4$.

Solution:

First we observe that

$$\frac{z-12}{z^2+z-6} = \frac{3}{z+3} - \frac{2}{z-2}.$$

We then use the geometric series expansions in each region after the following manipulations.

(a) In this region we have $\frac{1}{|z-1|} < 1$ and $\frac{|z-1|}{4} < 1$, so we need to observe that

$$\frac{2}{z-2} = \frac{2}{z-1}\frac{1}{1-\frac{1}{z-1}}, \tag{A}$$

and

$$\frac{3}{z+3} = \frac{3}{4}\frac{1}{1+\frac{z-1}{4}}. \tag{B}$$

(b) In addition to observation (B) above we need to observe that

$$\frac{2}{z-2} = \frac{-2}{1-(z-1)}. \tag{C}$$

(c) Here $\frac{4}{|z-1|} < 1$, so in particular we have $\frac{1}{|z-1|} < 1$. So in addition to observation (A) above we also observe that

$$\frac{3}{z+3} = \frac{3}{z-1}\frac{1}{1+\frac{4}{z-1}}.$$

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Q-4) Let C be the boundary of the rectangle with corners at the points $1,\ 1+i,\ -1+i,\ -1$. Evaluate the integral

$$I_n = \frac{1}{2\pi i}\int_C \frac{z^n}{z^2+z+1}\,dz,\ n = 0, 1, 2, \ldots.$$

Solution:

Let

$$f_n(z) = \frac{z^n}{z^2+z+1}.$$

First note that $z^2+z+1 = (z-a)(z-\bar{a})$ where $a = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\,i$. The only singularity of $f_n(z)$ inside C is a. Therefore

$$I_n = \text{Res}[f_n(z); a] = \frac{a^n}{a-\bar{a}} = \begin{cases} -\frac{i}{\sqrt{3}} & \text{if } n \equiv 0 \mod 3, \\ \frac{1}{2} + \frac{i}{2\sqrt{3}} & \text{if } n \equiv 1 \mod 3, \\ -\frac{1}{2} + \frac{i}{2\sqrt{3}} & \text{if } n \equiv 2 \mod 3. \end{cases}$$

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Q-5) Show that $\int_0^\infty \frac{dx}{1+x^{12}} = \frac{\pi}{6\sqrt{2-\sqrt{3}}} \approx 1.01151516.$

Solution:

The usual method works. Integrate from 0 to $R > 1$. Then along $|z| = R$ form $\theta = 0$ to $\theta = \pi/6$. And backwards along the line $z = re^{i\pi/6}$ where $0 \leq r \leq R$. Inside the loop there is only one singularity $z = e^{i\pi/12}$. The residue at that point is $-(1/12)\, e^{i\pi/12}$. Letting I denote the integral, we get

$$(1 - e^{i\pi/6})\, I = 2\pi i(-\frac{1}{12}\, e^{i\pi/12}).$$

Equating the real or imaginary parts of both sides we get

$$I = \frac{\pi}{12} \frac{1}{\sin \frac{\pi}{12}}.$$

Finally we calculate that

$$\sin \frac{\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$