MATH 202 Complex Analysis
Homework 3
Solution Key

1) Let ϕ_N be the stereographic projection of the Riemann sphere $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ onto the complex plane $x_3 = 0$, $(z = x_1 + ix_3)$. Let M_θ be the rotation of S around the x_1-axis, where $-\pi < \theta \leq \pi$. Show that

$$\phi_N \circ M_\theta \circ \phi_N^{-1}(z) = \begin{cases} \dfrac{z + i(\tan\frac{\theta}{2})}{i(\tan\frac{\theta}{2})z + 1} & -\pi < \theta < \pi \\ \\ \dfrac{1}{z} & \theta = \pi, \end{cases}$$

where the second stereographic projection is with respect to the new North pole of the sphere after the rotation by θ.

Solution:
We set for ease of notation $z = x + iy$. We use the following stereographic projection formulas.

$$\phi_N(U, V, W) = \left(\frac{U}{1 - W}, \frac{V}{1 - W}\right) \quad \text{and} \quad \phi_N^{-1}(z) = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right).$$

For any θ we have

$$M_\theta(U, V, W) = (U, V\cos\theta - W\sin\theta, V\sin\theta + W\cos\theta).$$

If we set

$$w(z) = \phi_N \circ M_\theta \circ \phi_N^{-1}(z) = X + iY,$$

then we have

$$X + iY = \left(\frac{2x}{|z|^2 + 1 - 2y\sin\theta - |z|^2\cos\theta + \cos\theta}, \frac{2y\cos\theta - |z|^2\sin\theta + \sin\theta}{|z|^2 + 1 - 2y\sin\theta - |z|^2\cos\theta + \cos\theta}\right).$$

We want to find $a, b, c, d \in \mathbb{C}$ such that

$$X + iY = \frac{az + b}{cz + d} = \frac{az + b}{cz + d} \cdot \frac{\bar{c}\bar{z} + \bar{d}}{\bar{c}\bar{z} + \bar{d}} = \frac{a\bar{c}|z|^2 + a\bar{d}z + b\bar{c}\bar{z} + b\bar{d}}{c\bar{c}|z|^2 + c\bar{d}z + \bar{c}d\bar{z} + d\bar{d}}$$

We now need to solve the following system for the unknowns a, b, c, d for all $z \in \mathbb{C}$.

$$\begin{aligned} |z|^2 + 1 - 2y\sin\theta - |z|^2\cos\theta + \cos\theta &= c\bar{c}|z|^2 + c\bar{d}z + \bar{c}d\bar{z} + d\bar{d} \\ 2x &= \operatorname{Re}(a\bar{c}|z|^2 + a\bar{d}z + b\bar{c}\bar{z} + b\bar{d}) \\ 2y\cos\theta - |z|^2\sin\theta + \sin\theta &= \operatorname{Im}(a\bar{c}|z|^2 + a\bar{d}z + b\bar{c}\bar{z} + b\bar{d}) \end{aligned}$$

At this point it helps if you set $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$, $d = d_1 + id_2$ and search for the real unknowns $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$.

We find the following solutions:

$$a_1 = -\sqrt{2}\cos\frac{\theta}{2}, \qquad a_2 = 0, \qquad b_1 = 0, \qquad b_2 = -\sqrt{2}\sin\frac{\theta}{2}$$

$$c_1 = 0, \qquad c_2 = \sqrt{2}\sin\frac{\theta}{2}, \qquad d_1 = -\sqrt{2}\cos\frac{\theta}{2}, \qquad d_2 = 0.$$

Thus we obtain the following Mobius transformation.

$$w(z) = \frac{(-\sqrt{2}\cos\frac{\theta}{2})z + (-i\sqrt{2}\sin\frac{\theta}{2})}{(-i\sqrt{2}\sin\frac{\theta}{2})z + (\sqrt{2}\cos\frac{\theta}{2})}$$

When $\theta \neq \pi$ we can divide each coefficient by the non-zero value $-\sqrt{2}\cos\frac{\theta}{2}$ to obtain

$$w(z) = \frac{z + i(\tan\frac{\theta}{2})}{i(\tan\frac{\theta}{2})z + 1}, \text{ when } \; -\pi < \theta < \pi.$$

When $\theta = \pi$, the Mobius transformation $w(z)$ becomes

$$w(z) = \frac{1}{z}, \text{ when } \; \theta = \pi.$$

2) Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}$. Let $T(z) = (z, z_2; z_3, z_4)$ be the cross-ratio morphism. For any $k \in \mathbb{C}$, can you find a Mobius transformation w such that $w(z_1) = k$, $w(z_2) = -k$, $w(z_3) = 1$, $w(z_4) = -1$? Can k be equal to i?

Solution:
Let $t \in \mathbb{C}$ be a complex number such that $t^2 = T(z_1$. Note $t \neq 0, 1$.

When $t \neq -1$, consider the Mobius transformation

$$S(z) = \frac{t+z}{t-z},$$

and set

$$k = -\frac{t+1}{t-1}.$$

Now check that

$$\begin{aligned} S(T(z_1)) &= S(\lambda) = k, \\ S(T(z_2)) &= S(1) = -k, \\ S(T(z_3)) &= S(0) = 1, \\ S(T(z_4)) &= S(\infty) = -1. \end{aligned}$$

When $t = -1$, consider the Mobius transformation

$$G(z) = \frac{i+z}{i-z}.$$

Then check that

$$\begin{aligned} G(T(z_2) &= G(1) = -i \\ G(T(z_3)) &= G(0) = 1 \\ G(T(z_4)) &= G(\infty) = -1. \end{aligned}$$

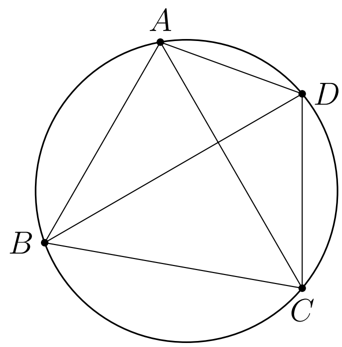
Now let z_1 be such that $T(z_1) = -1$. Then

$$G(T(z_1)) = G(-1) = i.$$

Hence $k = i$ is possible.

Obviously such a quadruple is easy to find. Let H be any Mobius transformation and set $z_1 = H(-1)$, $z_2 = H(1)$, $z_3 = H(0)$ and $z_4 = H(\infty)$. In this case $T = H^{-1}$ and $G \circ T$ takes z_1, z_2, z_3, z_4 to $i, -i, 1, -1$ as claimed.

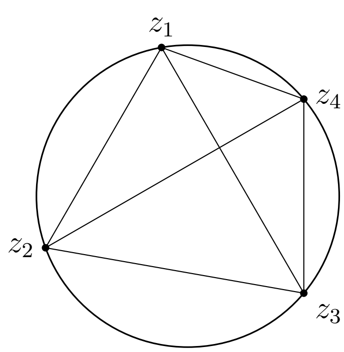
For this question consider **Ptolemy's Theorem**: A quadrilateral $ABCD$ is cyclic if and only if the sum of the products of the opposite sides equals the product of the diagonals. In other words, the points A, B, C, D lie on a circle if and only if $AC \cdot BD = AB \cdot DC + AD \cdot BC$.



3) Prove Ptolemy's theorem using the fact that the cross-ratio of four complex numbers is real if and only if the points lie on a circle.

Solution:

First we change our notation to comply with complex analysis. Let the points A, B, C and D be denoted by the complex numbers z_1, z_2, z_3 and z_4 in the complex plane. Assume further that the orientation of the points are as given in the figure below.



We want to prove that

$$|z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4| = |z_1 - z_3| \cdot |z_2 - z_4|$$

if and only if the points z_1, z_2, z_3, z_4 lie on a circle.

For the if part assume that the points lie on a circle.

Let T be the Mobius transformation such that

$$T(z_4) = \infty,\ T(z_3) = 0,\ T(z_2) = 1.$$

Then

$$T(z_1) = a > 1,$$

since Mobius transformations preserve circles, that is why $a \in \mathbb{R}$, and Mobius transformations preserve the orientation of points on the circle, that is why $a > 1$.

Since Mobius transformations also preserve cross-ratio, we have

$$\langle z_2, z_3, z_1, z_4 \rangle = \langle T(z_2), T(z_3), T(z_1), T(z_4) \rangle = \langle 1, 0, a, \infty \rangle = \frac{a-1}{a} > 0,$$

and

$$\langle z_2, z_1, z_3, z_4 \rangle = \langle T(z_2), T(z_1), T(z_3), T(z_4) \rangle = \langle 1, a, 0, \infty \rangle = \frac{1}{a} > 0.$$

Since $\langle z_2, z_3, z_1, z_4 \rangle$ and $\langle z_2, z_1, z_3, z_4 \rangle$ are positive and add up to 1, their absolute values also add up to 1.

Note that

$$|\langle z_2, z_3, z_1, z_4 \rangle| = \left| \frac{z_2 - z_1}{z_2 - z_4} \frac{z_3 - z_4}{z_3 - z_1} \right| \quad \text{and} \quad |\langle z_2, z_1, z_3, z_4 \rangle| = \left| \frac{z_2 - z_3}{z_2 - z_4} \frac{z_1 - z_4}{z_1 - z_3} \right|.$$

Therefore we have

$$\left| \frac{z_2 - z_1}{z_2 - z_4} \frac{z_3 - z_4}{z_3 - z_1} \right| + \left| \frac{z_2 - z_3}{z_2 - z_4} \frac{z_1 - z_4}{z_1 - z_3} \right| = 1.$$

Multiplying both sides by $|z_2 - z_4| \cdot |z_1 - z_3|$ we get the first part of Ptolemy's theorem,

$$|z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4| = |z_1 - z_3| \cdot |z_2 - z_4|.$$

For the only if part we again use the above Mobius transformation T except that this time we do not know the nature of a yet but we know that since we have

$$|z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4| = |z_1 - z_3| \cdot |z_2 - z_4|,$$

dividing both sides by $|z_1 - z_3| \cdot |z_2 - z_4|$ we get

$$\left| \frac{z_2 - z_1}{z_2 - z_4} \frac{z_3 - z_4}{z_3 - z_1} \right| + \left| \frac{z_2 - z_3}{z_2 - z_4} \frac{z_1 - z_4}{z_1 - z_3} \right| = 1,$$

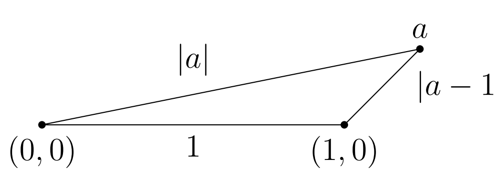
which is equivalent to

$$\left| \frac{a-1}{a} \right| + \left| \frac{1}{a} \right| = 1.$$

This in turn is equivalent to writing

$$|a - 1| + 1 = |a|.$$

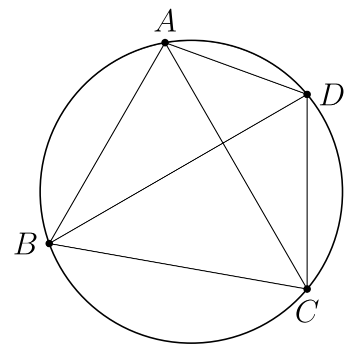
In the complex plane we have the triangle



We see that the triangle inequality holds as an equality for this triangle. hence the tree vertices of this triangle are collinear, i.e. a is real proving that the points z_1, z_2, z_3, z_4 lie on a circle. (In fact since T preserves orientaion we must have also $a > 1$ so the above arguments all fit into place.)

This completes the proof of Ptolemy's theorem using cross-ratio.

For this question consider **Ptolemy's Theorem**: A quadrilateral $ABCD$ is cyclic if and only if the sum of the products of the opposite sides equals the product of the diagonals. In other words, the points A, B, C, D lie on a circle if and only if $AC \cdot BD = AB \cdot DC + AD \cdot BC$.



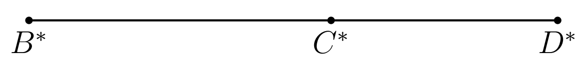
4) Let C be a circle with center at $a \in \mathbb{C}$ and radius $R > 0$. For any complex number z, let z^* denote its symmetric point with respect to C. Prove Ptolemy's theorem using the fact that for any two complex numbers z_1 and z_2, neither being a, we have $|z_1^* - z_2^*| = \dfrac{R^2}{|z_1 - a|\,|z_2 - a|}\,|z_1 - z_2|$.

Solution:

Notation: Throughout this solution we will treat the points as they are in $\mathbb{R}^2$ so that AB denotes the distance between the two points.

First assume that the given quadrilateral lies on a circle as in the above figure. Let K be a circle centered at A and containing the above circle in its interior. Let B^*, C^* and D^* be the symmetric points of B, C and D with respect to the circle K. Then the points B^*, C^* and D^* lie on a line and hence

$$B^*C^* + C^*D^* = B^*D^*.$$



Using the formula for symmetry we see that this equation gives us

$$\frac{BC}{AB \cdot AC} + \frac{CD}{AC \cdot AD} = \frac{BD}{AB \cdot AD}.$$

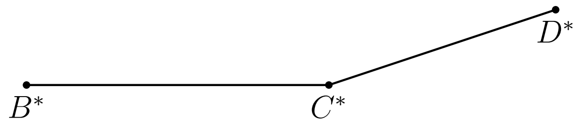
Multiplying both sides by $AB \cdot AC \cdot AD$ gives

$$AD \cdot BC + AB \cdot CD = AC \cdot BD,$$

which establishes one side of Ptolemy's theorem.

For the second part let S be the circle passing through the points A, B and C. Let K as before be a circle with center A and containing S in its interior.

Let B^*, C^* and D^* be the symmetric points of B, C and D with respect to the circle K. We now expect to see the following figure.



We are given that

$$AD \cdot BC + AB \cdot CD = AC \cdot BD.$$

Dividing both sides by $AB \cdot AC \cdot AD$ we get

$$\frac{BC}{AB \cdot AC} + \frac{CD}{AC \cdot AD} = \frac{BD}{AB \cdot AD}.$$

Using the formula for symmetry we see that this equation gives us

$$B^*C^* + C^*D^* = B^*D^*.$$

But this means that the points B^*, C^* and D^* lie on a line. This in turn means that the symmetry point D of D^* lies on the circle S, proving the other part of the theorem.