MATH 202 Complex Analysis
Homework 3
Due date: Due date: 28 December 2021 Tuesday Class Time

Show your work in reasonable detail. It is important that you explain your solution in a convincing way. I can but will not do mind reading!

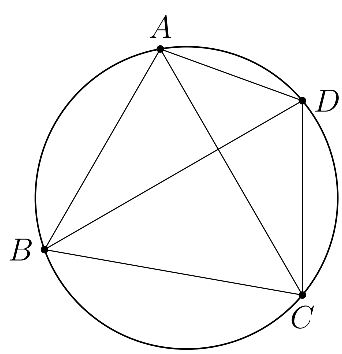
1) Let ϕ_N be the stereographic projection of the Riemann sphere $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ onto the complex plane $x_3 = 0$, $(z = x_1 + ix_2)$. Let M_θ be the rotation of S around the x_1-axis, where $-\pi < \theta \leq \pi$. Show that

$$\phi_N \circ M_\theta \circ \phi_N^{-1}(z) = \begin{cases} \dfrac{z + i(\tan\frac{\theta}{2})}{i(\tan\frac{\theta}{2})z + 1} & -\pi < \theta < \pi \\ \\ \dfrac{1}{z} & \theta = \pi, \end{cases}$$

where the second stereographic projection is with respect to the new North pole of the sphere after the rotation by θ.

2) Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}$. Let $T(z) = (z, z_2; z_3, z_4)$ be the cross-ratio morphism. For any $k \in \mathbb{C}$, can you find a Mobius transformation w such that $w(z_1) = k$, $w(z_2) = -k$, $w(z_3) = 1$, $w(z_4) = -1$? Can k be equal to i?

For the next two questions consider **Ptolemy's Theorem**: A quadrilateral $ABCD$ is cyclic if and only if the sum of the products of the opposite sides equals the product of the diagonals. In other words, the points A, B, C, D lie on a circle if and only if $AC \cdot BD = AB \cdot DC + AD \cdot BC$.



3) Prove Ptolemy's theorem using the fact that the cross-ratio of four complex numbers is real if and only if the points lie on a circle.

4) Let C be a circle with center at $a \in \mathbb{C}$ and radius $R > 0$. For any complex number z, let z^* denote its symmetric point with respect to C. Prove Ptolemy's theorem using the fact that for any two complex numbers z_1 and z_2, neither being a, we have $|z_1^* - z_2^*| = \dfrac{R^2}{|z_1 - a|\,|z_2 - a|}\,|z_1 - z_2|$.